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Two ways to compute Galois Cohomology using Lubin-Tate (φ, Γ)-modules, a Reciprocity Law and a Regulator Map

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Abstract

The main focus of the present thesis lays on general Lubin-Tate (φ, Γ)-modules. Before heading towards this theory, we discuss some general facts about monoid and continuous group cohomology as well as double complexes and limits of complexes.
After these preliminaries we first show as in the classical case that the category of étale (φ, Γ)-modules is equivalent to the category of Galois representations of the absolute Galois group of K with coefficients in $\mathcal{O}_L$, where $K|L$ and $L|\mathbb{Q}_p$ are finite extensions.
Using (φ, Γ)-modules, we then compute Iwasawa cohomology of such a representation and define a reciprocity map. Afterwards we compute the Galois cohomology groups using (φ, Γ)-modules. To do this, we construct two complexes of (φ, Γ)-modules whose cohomologies each coincide with the cohomology of the attached Galois representation. One of these two complexes is constructed by using the operator φ the other one by using the operator ψ. Finally, we construct a regulator map for an $\mathcal{O}_L^\times \times \mathbb{Z}_p$-extension of L.

Kurzdarstellung

Der Schwerpunkt der vorliegenden Arbeit liegt auf allgemeinen Lubin-Tate (φ, Γ)-Moduln. Bevor wir uns dieser Theorie widmen, behandeln wir einige allgemeine Aussagen über Monoid- und stetige Gruppenkohomologie sowie über Doppelkomplexe und Limites von Komplexen.
Nach diesen Vorbereitungen zeigen wir wie im klassischen Fall zunächst, dass die Kategorie der étalen (φ, Γ)-Moduln äquivalent ist zur Kategorie von Galoisdarstellungen der absoluten Galoisgruppe von K mit Koeffizienten in $\mathcal{O}_L$, wobei $K|L$ und $L|\mathbb{Q}_p$ endliche Erweiterungen sind.
Danach wird mithilfe von (φ, Γ)-Moduln Iwasawa Kohomologie einer Darstellung berechnet und eine Reziprozitätsabbildung definiert. Anschließend wird die Galoiskohomologie einer Darstellung mit (φ, Γ)-Moduln berechnet. Hierzu werden zwei Komplexe von (φ, Γ)-Moduln konstruiert, deren Kohomologie dann jeweils der Galoiskohomologie der zugehörigen Darstellung entspricht. Einer dieser beiden Komplexe wird zu dem Operator φ, der andere zu dem Operator ψ gebildet. Den Abschluss der Arbeit bildet die Konstruktion einer Regulatorabbildung für eine $\mathcal{O}_L^\times \times \mathbb{Z}_p$-Erweiterung von L.

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Chapter 1

Introduction

One of the most interesting and most studied objects in algebraic number theory are absolute Galois groups. Since these groups are far away from being easy to understand, mathematicians discovered lots of paths to describe them in numerous ways and to reveal their secrets. An often used tool are the representations of these groups and the corresponding cohomology groups.

Let p be a prime number and fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be the p-adic completion of $\overline{\mathbb{Q}_p}$. Assume that all algebraic extensions of $\mathbb{Q}_p$ are inside $\overline{\mathbb{Q}_p}$. Furthermore, $L|\mathbb{Q}_p$ be a finite extension, $\mathcal{O}_L$ its ring of integers and π_L be a prime element of $\mathcal{O}_L$.

The p-adic Hodge theory, studies representations of (infinite) Galois groups with values in L or its ring of integers $\mathcal{O}_L$. Fontaine then established a new sight on these p-adic Galois representations as he showed that étale (φ, Γ)-modules are equivalent to p-adic Galois representations (cf. [FO10, Theorem 4.22, p.82]) over $\mathbb{Q}_p$ (or $\mathbb{Z}_p$). One great benefit of this construction is that (φ, Γ)-modules are objects of (semi-)linear algebra and therefore relatively easy to understand. However, this comes at the cost of a more complicated coefficient ring. For the construction of this coefficient ring, Fontaine used the cyclotomic extension of $\mathbb{Q}_p$. In a natural way then there arose two questions: First, if there is a similar construction of (φ, Γ)-modules for Lubin-Tate extensions (since the cyclotomic extension is a special case of Lubin-Tate extensions) and second, if there is a category of (φ, Γ)-modules which is equivalent to Galois representations over a finite extension of $\mathbb{Q}_p$ or its integers. In 2009 Kisin and Ren answered both questions with "yes" (cf. [KR09, Theorem (1.6), p. 446]) and in 2017 Schneider gave a proof in full detail (cf. [Sch17, Theorem 3.3.10, p. 134]). While Schneider's proof covers the case of representations of G_L (the absolute Galois group of L) with values in $\mathcal{O}_L$, Kisin and Ren stated the result also for subgroups of G_L, i.e.

absolute Galois groups of finite extensions of L. In this work, we generalize Schneider's proof to the latter case, i.e. to the case where the considered representations are the representations of G_K, where $K|L$ is finite, with values in $\mathcal{O}_L$. Before giving the exact statement of the theorem, we should say a word about what Γ is and about the coefficient ring of our (φ, Γ)-modules. We start with Γ. We are interested in certain algebraic extension of L with Galois group isomorphic to $\mathcal{O}_L^\times$. We fix such an extension and denote it by L_∞. In the classical theory this is $\mathbb{Q}_p(\mu_{p^\infty})$, i.e. the extension of all p^n-th roots of unity. In the general case, L_∞ is the union of the extensions generated by the roots of the powers of so called Frobenius power series. These are power series with coefficients in $\mathcal{O}_L$ such that they are congruent to $\pi_L X$ modulo degree 2 and congruent to X^{q_L} module $\pi_L \mathcal{O}_L[\![X]\!]$, where q_L is the cardinality of $\mathcal{O}_L/(\pi_L)$. The classical case fits also in this theory since $(X-1)^p - 1$ fulfills the above requirements and if α is a root of its n-th power, then $\alpha + 1$ is a p^n-th root of unity, i.e. the field extension generated by the roots of its n-th power coincides with the extension of p^n-th roots of unity. Anyway, the Galois group of $L_\infty|L$ is denoted by Γ_L and the one of $\overline{\mathbb{Q}_p}|L_\infty$ by H_L. Furthermore, we let $K|L$ be a finite extension and denote let $K_\infty := KL_\infty$. We then denote by Γ_K the Galois group of $K_\infty|K$ and by H_K the one of $\overline{\mathbb{Q}_p}|K_\infty$.
For the coefficient ring, let $\mathbb{C}_p^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ be the tilt of $\mathbb{C}_p$ and denote by $W(\mathbb{C}_p^\flat)_L$ the ramified Witt vectors with respect to L. Then we start with the ring

$$\mathbf{A}_L \cong \varprojlim_{n \in \mathbb{N}} \mathcal{O}_L/\pi_L^n \mathcal{O}_L((X))$$

which can be realized as a subring of $W(\mathbb{C}_p^\flat)_L$ (cf. [Sch17, p. 94]). Next, we consider $\mathbf{A} = (\mathbf{A}_L^{\mathrm{nr}})^\wedge$, where $\mathbf{A}_L^{\mathrm{nr}}$ is the maximal unramified extension of $\mathbf{A}_L$ inside $W(\mathbb{C}_p^\flat)_L$ and $()^\wedge$ denotes the completion with respect to the p-adic topology. Since $\mathbb{C}_p^\flat$ has characteristic p, it has a Frobenius homomorphism and since G_L acts on $\mathbb{C}_p$ it also acts on $\mathbb{C}_p^\flat$. By functoriality, $W(\mathbb{C}_p^\flat)_L$ then also has a Frobenius and an action from G_L. Both carry over to $\mathbf{A}$. Therefore we can define the ring $\mathbf{A}_{K|L} := \mathbf{A}^{H_K}$ which then also has a Frobenius, denoted by $\varphi_{K|L}$, and an action from $\Gamma_K = G_K/H_K$. We should also say a word about topologies. Both, the endomorphism $\varphi_{K|L}$ and the action from Γ_K are continuous with respect to the so called weak topology on $\mathbf{A}_{K|L}$. This is the subspace topology from $W(\mathbb{C}_p^\flat)_L$, where $\mathbb{C}_p^\flat$ carries the topology of the projective limit and each $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ carries the discrete topology. $W(\mathbb{C}_p^\flat)_L$ then carries product topology. Under the above isomorphism for $\mathbf{A}_L$, the weak topology of $\mathbf{A}_L$ is generated by the sets (c.f. [Sch17, p. 79] and [Sch17, Proposition 2.1.16,

p. 95–96])

$$X^n\mathcal{O}_L[\![X]\!] + \pi_L^n\mathbf{A}_L.$$

The weak topology on $\mathbf{A}_{K|L}$ has the same structure (cf. Corollary 3.3.4) and it coincides with the weak topology considered as $\mathbf{A}_L$-module (cf. Proposition 3.3.5). For this theory, we cannot work with the p-adic topology, since, for example, the G_L-action on $W(\mathbb{C}_p^\flat)_L$ is not continuous with respect to the p-adic topology on $W(\mathbb{C}_p^\flat)_L$ (cf. [Sch09, Bemerkung 3.2.11, p. 106]). In Section 3.3 we study weak topologies on both, the coefficient ring $\mathbf{A}_{K|L}$ and its finitely generated modules in detail.
A (φ,Γ)-module over $\mathbf{A}_{K|L}$ then is a finitely generated $\mathbf{A}_{K|L}$-module M together with a semilinear action from Γ_K and a $\varphi_{K|L}$-semilinear endomorphism φ_M. M is called étale if the linearized homomorphism

$$\mathbf{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} M \longrightarrow M,\ a\otimes m \longmapsto a\varphi_M(m)$$

is bijective. Here ${}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}}$ means that $\mathbf{A}_{K|L}$ is considered as $\mathbf{A}_{K|L}$-module via $\varphi_{K|L}$. The theorem then reads (cf. Theorem 3.9.1):

Theorem A.
The categories $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ *and* $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ *are equivalent to each other. The equivalence is given by the quasi invers functors*

$$\begin{aligned}\mathcal{M}_{K|L}\colon \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K) &\longrightarrow \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})\\ V &\longmapsto (\mathbf{A}\otimes_{\mathcal{O}_L} V)^{H_K}\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}_{K|L}\colon \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L}) &\longrightarrow \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)\\ M &\longmapsto \left(\mathbf{A}\otimes_{\mathbf{A}_{K|L}} M\right)^{\mathrm{Fr}\otimes\varphi_M=1}.\end{aligned}$$

This is the generalization to finite extensions of $\mathbb{Q}_p$ and to Lubin-Tate (φ,Γ)-modules of Fontaine's original equivalence of categories. In the above equivalence one can see that the change to a subgroup on the side of Galois representations translates into a change of the coefficient ring, the involved group Γ and the endomorphism φ on the side of (φ,Γ)-modules. In Section 3.4 we study the structure of the coefficient ring for unramified extensions and in Section 3.5 for general extensions.

In the following chapters, we use these general (φ,Γ)-modules to calculate Iwasawa and continuous cohomology for a representation of G_K with coefficients in $\mathcal{O}_L$, establish a reciprocity law, which generalizes the corresponding reciprocity law from Schneider and Venjakob (c.f. [SV15, Theorem 6.2, p. 32]) and construct a regulator

map, which interpolates the regulator maps from [SV19, Section 3.1, p. 71–74].
In Chapter 4 we start with taking a closer look at [SV15]. Using the original result of Coleman (cf. [Col79, Thm. A, p. 92]) allows us to generalize Schneider's and Venjakob's work to the case of a finite and unramified extension K of L. Since $K|L$ is unramified, it as a Galois extension with cyclic Galois group, generated by the Frobenius $\sigma_{K|L}$, which is a lift from the Frobenius of the corresponding residue class field extension. Thankfully, their original work is very detailed and so our main task was to study what the input from the Frobenius is. The short version is: The results do not really change, sometimes there is a shift by the Frobenius, but that's exactly what one needs to establish these results in this bigger generality. That the Frobenius is involved is a consequence of Proposition 3.4.6. There we see that we have

$$\varphi_{K|L} = \sigma_{K|L} \circ \varphi_L$$

on $\mathbf{A}_{K|L}$, $\varphi_{K|L}$ is the φ-operator of the (φ, Γ)-modules over $\mathbf{A}_{K|L}$ and φ_L the one of the (φ, Γ)-modules over $\mathbf{A}_L$. We then also introduce a ψ operator by

$$\psi_{K|L} = \frac{1}{\pi_L} \varphi_{K|L}^{-1} \circ \mathrm{Tr},$$

where Tr is the trace map of the finite extension $\mathbf{B}_{K|L}|\varphi_{K|L}(\mathbf{B}_{K|L})$. One of the results in this chapter then is the relation of this ψ-operator to the Iwasawa cohomology of an $\mathcal{O}_L$-representation of G_K (cf. Theorem 4.3.13):

Theorem B.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$, $\tau = \chi_{\mathrm{cyc}}\chi_{\mathrm{LT}}^{-1}$ and $\psi = \psi_{\mathcal{M}_{K|L}(V(\tau^{-1}))}$. Then we have an exact sequence

$$0 \to H^1_{\mathrm{Iw}}(K_\infty|K, V) \to \mathcal{M}_{K|L}(V(\tau^{-1})) \xrightarrow{\psi - \mathrm{id}} \mathcal{M}_{K|L}(V(\tau^{-1})) \to H^2_{\mathrm{Iw}}(K_\infty|K, V) \to 0,$$

which is functorial in V.
Furthermore, each occurring map is continuous and $\mathcal{O}_L[\![\Gamma_K]\!]$-equivariant.

Here we have $\tau = \chi_{\mathrm{cyc}}\chi_{\mathrm{LT}}^{-1}$, where χ_{cyc} denotes the cyclotomic character and $\chi_{\mathrm{LT}} \colon \Gamma_L \xrightarrow{\cong} \mathcal{O}_L^\times$ the Lubin-Tate character. The chapter then concludes in the following reciprocity law (cf. Theorem 4.4.2).

Theorem C.
The following diagram is commutative:

$$\begin{tikzcd}
(\varprojlim_n K_n^\times) \otimes_{\mathbb{Z}_p} T^* \arrow[rr, "-\kappa\otimes\mathrm{id}_{T^*}", "\cong"'] \arrow[dr, "\nabla"'] & & H^1_{\mathrm{Iw}}(K_\infty|K, \mathcal{O}_L(\tau)) \arrow[dl, "\cong"', "\mathrm{Exp}^*"] \\
& (\mathbf{A}_{K|L})^{\psi=1}. &
\end{tikzcd}$$

Here, Exp^* denotes the homomorphism induced from

$$H^1_{\mathrm{Iw}}(K_\infty|K, \mathcal{O}_L(\tau)) \to \mathcal{M}_{K|L}(\mathcal{O}_L) = \mathbf{A}_{K|L}$$

from the above Theorem B (respectively from Theorem 4.3.13), T^* is the representation module of χ_{LT}^{-1}, i.e. T^* is isomorphic to $\mathcal{O}_L$ as $\mathcal{O}_L$-module and carries an action from Γ_K by $\gamma \cdot t = \chi_{\mathrm{LT}}(\gamma)^{-1}t$. One can proof that there is a natural isomorphism $H^i_{\mathrm{Iw}}(K_\infty|K, V \otimes_{\mathcal{O}_L} T^*) \cong H^i_{\mathrm{Iw}}(K_\infty|K, V) \otimes_{\mathcal{O}_L} T^*$ (cf. Remark 4.3.10). Together with $V \otimes_{\mathcal{O}_L} T^* \cong V(\chi_{\mathrm{LT}}^{-1})$ and the Kummer isomorphism $\varprojlim_n K_n^\times \cong H^1_{\mathrm{Iw}}(K_\infty|K, \mathbb{Z}_p(1))$ this then induces the horizontal homomorphism in the above diagram. Note that $\varprojlim_n K_n^\times$ denotes the norm field of K_∞, i.e. the projective limit is build with respect to the norm maps $K_{n+1} \to K_n$. Then $\varprojlim_n K_n^\times$ is the multiplicative group of a field (cf. [Win83, 2.1.3 Théorème, p. 65–66]). In the key lemma (cf. Lemma 4.4.4) of the above theorem a Frobenius is involved again, which does not appear in the original work. We then end the section (and the chapter) by explaining at which part of the proofs this Frobenius comes in and that it has to be there.

In Chapter 5 we study how to compute Galois cohomology using (φ, Γ)-modules. In the classical theory, this is well known and is one of the benefits of (φ, Γ)-modules: If V is a $\mathbb{Z}_p$-representation of G_K, then the complex

$$0 \longrightarrow \mathcal{M}_{K|\mathbb{Q}_p}(V) \xrightarrow{(f-1,\gamma-1)} \mathcal{M}_{K|\mathbb{Q}_p}(V) \oplus \mathcal{M}_{K|\mathbb{Q}_p}(V) \xrightarrow{(\gamma-1)\mathrm{pr}_1-(f-1)\mathrm{pr}_2} \mathcal{M}_{K|\mathbb{Q}_p}(V) \longrightarrow 0$$

computes the group cohomology of G_K with values in V, where f can be both, φ or its left-inverse ψ and where γ is a topological generator of Γ (cf. eg. [Col04, Theorem 5.2.2., p. 93–94] and [Col04, Theorem 5.3.15, p. 103–104]). These complexes are often called **Herr complexes**. Moreover, the complexes for φ and ψ are quasi-isomorphic (cf. [Col04, Proposition 5.3.14, p. 103]). While one has the exact same results for Lubin-Tate (φ, Γ)-modules over $\mathbf{A}_{\mathbb{Q}_p}$ (cf. [Kup15, Satz 2.20, p. 41–42] respectively [Kup15, Satz 2.26, p. 48] and [Kup15, Satz 2.27, p. 48]) one could not expect that this is also true for Lubin-Tate (φ, Γ)-modules corresponding to representations over a finite extension of $\mathbb{Q}_p$, since in this case the ψ-operator is no longer a left inverse to φ (cf. Remark 4.2.3). In summer 2019 Aribam and Kwatra published a partial result. They showed, that a generalized Herr complex with respect to φ computes the Galois cohomology of a torsion representation with coefficient ring $\mathcal{O}_K$, where $K|\mathbb{Q}_p$ is finite (cf. [AK19, Theorem 3.16, p. 10–11]). In this thesis we go a step further and prove that for an arbitrary finitely generated $\mathcal{O}_L$-representation V of G_K there is a complex of the corresponding (φ, Γ)-module, of which the cohomology is exactly the continuous group cohomology of G_K with coefficients in V. In our proof, we followed the idea of Scholl in [Sch06, Theorem 2.2.1, p. 702–705], generalized his proof to our setting and added all the details. By $C^\bullet_{\text{cts}}(G, A)$ we denote the continuous cochain complex of a profinite group G with values in the abelian group A. Furthermore, for $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ we denote by $\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M)$ the total complex of the double complex

$$C^\bullet_{\text{cts}}(\Gamma_K, M)) \xrightarrow{C^\bullet_{\text{cts}}(\Gamma_K, \varphi_M)-\text{id}} C^\bullet_{\text{cts}}(\Gamma_K, M)$$

and by $\mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M)$ its cohomology. The exact statement of the theorem then is (cf. Theorem 5.1.11):

Theorem D.
Let $V \in \mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ and set $M = \mathcal{M}_{K|L}(V)$. Then there are isomorphisms

$$H^*_{\text{cts}}(G_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M),$$

$$H^*_{\text{cts}}(H_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(M).$$

These isomorphisms are functorial in V and compatible with restriction and corestriction.

The idea of the proof is the following: First, we show that for discrete V the complexes $\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_n)$ and $C^\bullet_{\text{cts}}(G_K, V)$ are quasi isomorphic to the complex $\mathcal{C}^\bullet_{\text{Fr}}(G_K, \mathbf{A}/\omega^n_\phi \mathbf{A}^+ \otimes_{\mathcal{O}_L} V/\pi^m_L V)$. Here, the latter complex is defined in an analogous way as above and $M_n = \mathcal{M}_{K|L}(V)/(\omega^n_\phi \mathbf{A}^+ \otimes_{\mathcal{O}_L} V)^{H_K}$. These quasi isomorphisms are induced by the short exact sequence

$$0 \longrightarrow V \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V \xrightarrow{\text{Fr}\otimes\text{id}_V - \text{id}} \mathbf{A} \otimes_{\mathcal{O}_L} V \longrightarrow 0$$

respectively by the canonical inclusion $\mathcal{M}_{K|L}(V) \hookrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V$. In particular, both quasi isomorphisms have target $\mathcal{C}^\bullet_{\text{Fr}}(G_K, \mathbf{A}/\omega^n_\phi \mathbf{A}^+ \otimes_{\mathcal{O}_L} V/\pi^m_L V)$. After that, we take projective limits with respect to m and n and check that everything behaves well.

In the second part of Chapter 5, we head towards the computation of the Galois cohomology using the ψ-operator. Since φ and ψ are related to each other under Pontrjagin duality (cf. [SV15, Remark 5.6, p. 27]), it seems to be the correct way, to dualize the complex of φ. In a first attempt we tried to imitate the methods of Herr (cf. [Her01, Lemme 5.6, p. 333]) to establish a quasi isomorphism between the complexes of (φ, Γ)-modules related to φ and ψ using Tate duality. This approach requires to show that all the differentials of the φ-Herr complex have closed image, which implies that they are strict which then implies that the cohomology groups of the dualized complex coincide with the dual of the cohomology groups of the complex we started with. In his original work, Herr checked that the differentials have closed image for each differential separately (cf. [Her01, p. 334]). Unfortunately, in the general case we have to deal with direct products of Herr's differentials and modules and it is no longer clear, that the differentials have closed image.
Our second attempt then was successful. Here we imitated results of Nekovář (cf. [Nek07, Sections (8.2) and (8.3), p. 157–160]) to replace the complex $C^\bullet_{\text{cts}}(H_K, A)$ with a complex $C^\bullet_{\text{cts}}(G_K, F_{\Gamma_K}(A))$ of $\Lambda_K = \mathcal{O}_L[\![\Gamma_K]\!]$-modules, where $A = V^\vee$ is the dual of some G_K-representation. Here "replace" means, that the two complexes are quasi isomorphic (cf. Proposition 5.2.21). This then has the advantage that we can apply the Mattlis dual $\overline{\mathrm{D}_K} = \mathrm{Hom}_{\Lambda_K}(-, \Lambda_K^\vee)$ to this complex. Nekovář proved that this dualized complex is quasi isomorphic to a complex computing the Iwasawa cohomology (cf. Lemma 5.2.44). We then finally check, that the complex related to ψ is quasi isomorphic to this dualized complex. To do this, we use Theorem 4.3.13 and therefore we have to assume that $K|L$ is unramified. Using again a result of Nekovář , we then get the following statement (cf. Theorem 5.2.52).

Theorem E.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and let $K \subseteq K' \subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ is isomorphic to some $\mathbb{Z}_p^r$. Then we have an isomorphism in $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma(\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1})) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma^\bullet_{\text{cts}}(G_{K'}, T).$$

Here we use the following notation. If $\mathcal{C}^\bullet$ is a bounded below complex of abelian groups (or of R-modules for a suitable ring R), then we denote by $\mathbf{R}\Gamma(\mathcal{C}^\bullet)$ the same complex viewed as object in the derived category $\mathbf{D}^{\mathrm{b}}(\mathbf{Ab})$ (respectively in $\mathbf{D}^{\mathrm{b}}(R\text{-}\mathbf{Mod})$). Furthermore, by manipulating the representation on the right hand side we can replace the above complex of cochains of $G_{K'}$ by a complex of cochains of G_K (cf. Corollary 5.2.54). To be more precise, Shapiro's Lemma induces an isomorphism in $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_{K'}, T) \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/G_{K'}]),$$

where G_K acts diagonal on $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/G_{K'}]$.

In Chapter 6, which is the last chapter of this thesis, we generalize the regulator map of Loeffler and Zerbes of [LZ14a] to the case of a general Lubin-Tate extension. For this, let $F|L$ be a finite, unramified extension, F_∞ the unique unramified $\mathbb{Z}_p$-extension of F and $\Upsilon = \mathrm{Gal}(F_\infty|F)$. We then fix some additional notation, introduce more of Fontaine's period rings and introduce crystalline and analytic representations as well as explain the notion of $\mathbb{Q}_p$- and L-analytic functions and their continuous duals, the distributions. We denote the $\mathbb{Q}_p$-analytic distributions of Υ with values in $\mathbb{C}_p$ by $D_{\mathbb{Q}_p}(\Upsilon, \mathbb{C}_p)$ and the L-analytic distributions of Γ_L with vales in $\mathbb{C}_p$ by $D_L(\Gamma_L, \mathbb{C}_p)$. Afterwards, we recall a result from [Pic18] about the existence of an integral normal basis generator (i.e. an integral element whose powers are a normal basis), which says, that for finite and unramified Galois extensions over L such an integral normal basis generator always exists. Unfortunately we have to assume $p \neq 2$ for this. Then we introduce the Yager module, which turns out to be a free rank 1-module over the Iwasawa algebra of Υ over $\mathcal{O}_F$ and subsequently we introduce Wach modules and show that a Wach module over $\mathcal{O}_{F_\infty}$ is linked to the Wach module over $\mathcal{O}_F$ and the Yager module. Then we are almost prepared to introduce the Regulator map but we still have to face one detail, which is not known to be true in the general Lubin-Tate setting. This is, if there exists an $\mathbf{A}_L$-basis $(u_1, \ldots, u_n)$ of $\varphi_L(\mathbf{A}_L)$ such that $\psi_L(u_i) = \delta_{1i}$. But an analogous result is known for the Robba ring and its plus part. This together with results from [SV19] then allows us to introduce a regulator map similar to the one of Loeffler and Zerbes. Roughly (the full statement involves to many details for an introduction - for the full statement see Theorem 6.6.7) the theorem then is.

Theorem F.
Let $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_L)$ and $V = T[1/\pi_L]$ with nonnegative Hodge-Tate weights and such that T has no quotient isomorphic to the trivial representation. Then we have a regulator map

$$\mathcal{L}^{\Gamma_L,\Upsilon}_V \colon H^1_{\mathrm{Iw}}(F_\infty L_\infty|L, T) \longrightarrow D_L(\Gamma_L, \mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})).$$

This map interpolates the corresponding regulator maps for all finite intermediate fields of $F_\infty|F$.

Chapter 2

Preliminaries

By $\mathbb{N}$ we denote the natural numbers starting with 1 and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a homomorphism $f\colon A \to B$ we denote by $\ker(f)$ its kernel, by $\mathrm{im}(f)$ its image and by $\mathrm{coker}(f)$ its cokernel.

2.1 On Continuous Group Cohomology

First, we want to recall some basic facts from topology.
For topological spaces X, Y we endow the set of continuous maps $\mathrm{Map}_{\mathrm{cts}}(X, Y)$ always with the compact open topology (cf. [Bou89b, Definition 1, Chapter X §3.4, p. 301]). Note, that in this topology $\mathrm{Map}_{\mathrm{cts}}(X, Y)$ is a Hausdorff space if Y is (cf. [Bou89b, Remarks (1), Chapter X §3.4, p. 301–302]). For $K \subseteq X$ compact and $U \subseteq Y$ open denote by $\mathrm{M}(K, U)$ the set of all $f \in \mathrm{Map}_{\mathrm{cts}}(X, Y)$ with $f(K) \subseteq U$.

Theorem 2.1.1.
Let X, Y, Z be topological spaces and $f\colon X \times Y \to Z$ a map. If f is continuous, then also the map $\widetilde{f}\colon X \to \mathrm{Map}_{\mathrm{cts}}(Y, Z)$ is continuous, where $(\widetilde{f}(x))(y) = f(x, y)$.
If $\widetilde{f}$ is continuous and Y is locally compact, then also f is continuous.

Proof.
This is [Bou89b, Theorem 3, Chapter X §3.4, p. 302–303]). □

Corollary 2.1.2.
Let X and Y be topological spaces and X locally compact. Then the evaluation map $\mathrm{ev}\colon X \times \mathrm{Map}_{\mathrm{cts}}(X, Y) \to Y, (x, f) \mapsto f(x)$ is continuous.

Proof.
Since X is locally compact Theorem 2.1.1 says that the continuity of ev is equivalent

to the continuity of

$$\widetilde{\mathrm{ev}}\colon \mathrm{Map}_{\mathrm{cts}}(X,Y) \to \mathrm{Map}_{\mathrm{cts}}(X,Y),\ (\widetilde{\mathrm{ev}}(f))(x) = \mathrm{ev}(x,f) = f(x).$$

But $\widetilde{\mathrm{ev}}$ is the identity of $\mathrm{Map}_{\mathrm{cts}}(X,Y)$ and therefore continuous. □

Proposition 2.1.3.
Let X, Y, Z be topological spaces, X Hausdorff and Y locally compact. Then there is a homeomorphism

$$\mathrm{Map}_{\mathrm{cts}}(X \times Y, Z) \to \mathrm{Map}_{\mathrm{cts}}(X, \mathrm{Map}_{\mathrm{cts}}(Y,Z))$$

which is given by the restriction of the canonical bijection $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z))$.

Proof.
This is [Bou89b, Corollary 2 to Theorem 3, Chapter X §3.4, p. 303–304]. □

Definition 2.1.4.
Let M be a monoid and A an abelian group. We say that M **acts on** A if there is a map

$$\cdot\colon M \times A \longrightarrow A$$

which fulfills the following conditions:

1. $1_M \cdot a = a$ for all $a \in A$.
2. $m \cdot (a+b) = m \cdot a + m \cdot b$ for all $m \in M$ and $a, b \in A$.
3. $(mn) \cdot a = m \cdot (n \cdot a)$ for all $m, n \in M$ and $a \in A$.

If M is a topological monoid and A is a topological Hausdorff abelian group, then we say that an action is **continuous** if the above map "$\cdot$" is continuous.

Proposition 2.1.5.
Let G, H, A be topological groups such that H is locally compact and A is abelian and Hausdorff. Let furthermore G act continuously on both H and A. Then G also acts continuously on $\mathrm{Map}_{\mathrm{cts}}(H,A)$, where for $\sigma \in G$ and $f \in \mathrm{Map}_{\mathrm{cts}}(H,A)$ the action is given by $(\sigma \cdot f)(h) = \sigma(f(\sigma^{-1}(h)))$.

Proof.
First we should check that the action is well defined, i.e. we show that for $\sigma \in G$ and

$f \in \mathrm{Map}_{\mathrm{cts}}(H, A)$ we have $\sigma \cdot f \in \mathrm{Map}_{\mathrm{cts}}(H, A)$. So, let $\sigma \in G$ and $f \in \mathrm{Map}_{\mathrm{cts}}(H, A)$. Then the map $\sigma \cdot f$ can be written as composite of the following maps:

$$\begin{array}{cccccccc} H & \longrightarrow & H & \longrightarrow & A & \longrightarrow & A \\ h & \longmapsto & \sigma^{-1}(h) & & & & \\ & & h & \longmapsto & f(h) & & \\ & & & & a & \longmapsto & \sigma(a). \end{array}$$

The first of these maps is continuous since inversion in G is continuous and G acts continuously on H. The second one is f and therefore continuous. The last one is continuous since G acts continuously on A. So, in conclusion $\sigma \cdot f$ is a continuous map from H to A.
For the continuity of the group action, we have to show that the map

$$G \times \mathrm{Map}_{\mathrm{cts}}(H, A) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(H, A),\ (\sigma, f) \longmapsto \sigma \cdot f$$

is continuous. Since H is assumed to be locally compact this is equivalent to the continuity of the map

$$G \times H \times \mathrm{Map}_{\mathrm{cts}}(H, A) \longrightarrow A,\ (\sigma, h, f) \longmapsto \sigma(f(\sigma^{-1}(h)))$$

(cf. Theorem 2.1.1). This last map can be written as the composite of the following maps:

$$\begin{array}{ccccccc} G \times H \times \mathrm{Map}_{\mathrm{cts}}(H, A) & \longrightarrow & G \times H \times \mathrm{Map}_{\mathrm{cts}}(H, A) & \longrightarrow & G \times A & \longrightarrow & A \\ (\sigma, h, f) & \longmapsto & (\sigma, \sigma^{-1}(h), f) & & & & \\ & & (\sigma, h, f) & \longmapsto & (\sigma, f(h)) & & \\ & & & & (\sigma, a) & \longmapsto & \sigma(a). \end{array}$$

The first of these maps is continuous since G is a topological group (therefore inversion is continuous) and G acts continuously on H. The second map is continuous since evaluating functions with a locally compact domain are continuous (cf. Corollary 2.1.2). The last map is continuous since G acts continuously on A. □

Definition 2.1.6.
Let G be a profinite group and A an abelian topological group on which G acts continuously. We say that A is **G-induced** if there exists an abelian topological group B together with a continuous action of G, such that $A = \mathrm{Map}_{\mathrm{cts}}(G, B)$.

Lemma 2.1.7.
Let G be a profinite group and A an abelian topological group on which G acts continuously. Then the complex

$$0 \longrightarrow A \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G, A) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^2, A) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^3, A) \longrightarrow \cdots$$

is exact.

Proof.
At [NSW15, (1.2.1) Proposition, Chapter I §2, p. 12–13] is a proof for discrete group cohomology. For continuous cohomology it's literally the same, but one should check (in both cases) that the maps

$$\begin{aligned} D^n \colon \mathrm{Map}_{\mathrm{cts}}(G^{n+2}, A) &\longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A), \\ x &\longmapsto [(\sigma_0, \dots, \sigma_n) \mapsto x(1, \sigma_0, \dots, \sigma_n)] \end{aligned}$$

are well defined, namely that $D^n(x)$ for $x \in \mathrm{Map}_{\mathrm{cts}}(G^{n+2}, A)$ is continuous. For this, let $U \subseteq A$ be open and $x \in \mathrm{Map}_{\mathrm{cts}}(G^{n+2}, A)$. Then $x^{-1}(U) \subseteq G^{n+2}$ is open and so is $x^{-1}(U) \cap \{1\} \times G^{n+1}$ in $\{1\} \times G^{n+1}$. Since the canonical projection $\eta \colon \{1\} \times G^{n+1} \to G^{n+1}$ is open, the set $\eta(x^{-1}(U) \cap \{1\} \times G^{n+1})$ is open in G^{n+1}. The claim is, that this is exactly $(D^n x)^{-1}(U)$, which then proves the continuity of $D^n x$.
To see this, let $(\sigma_0, \dots, \sigma_n) \in (D^n x)^{-1}(U)$. Since $D^n x(\sigma_0, \dots, \sigma_n) = x(1, \sigma_0, \dots, \sigma_n)$ it follows $(1, \sigma_0, \dots, \sigma_n) \in x^{-1}(U)$ and we have $\eta(1, \sigma_0, \dots, \sigma_n) = (\sigma_0, \dots, \sigma_n)$, which means that $(\sigma_0, \dots, \sigma_n) \in \eta(x^{-1}(U) \cap \{1\} \times G^{n+1})$.
Conversely let $(\sigma_0, \dots, \sigma_n) \in \eta(x^{-1}(U) \cap \{1\} \times G^{n+1})$. Then we clearly have $(1, \sigma_0, \dots, \sigma_n) \in x^{-1}(U)$. Since $x(1, \sigma_0, \dots, \sigma_n) = D^n x(\sigma_1, \dots, \sigma_n)$ we immediately obtain $(\sigma_0, \dots, \sigma_n) \in (D^n x)^{-1}(U)$. □

Remark 2.1.8.
We want to recall the continuous standard resolution from [NSW15, Chapter II, §7, p. 136-137] *and fix the notation.*
Let G be a profinite group and A a topological Hausdorff abelian group with a continuous action from G. Let for $n \in \mathbb{N}_0$

$$X^n_{\mathrm{cts}}(G, A) \coloneqq \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$$

and $\partial^n_{\mathrm{cts}} \colon X^{n-1}_{\mathrm{cts}} \to X^n_{\mathrm{cts}}$ *be the differential , which is given by*

$$\partial^n_{\mathrm{cts}}(x)(\sigma_0, \dots, \sigma_n) = \sum_{i=0}^{n} (-1)^i x(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n),$$

where "$\hat{\ }$" means that the corresponding element is omitted. Furthermore, we denote by $X^\bullet_{\mathrm{cts}}(G, A)$ *the corresponding complex, i.e.*

$$X^\bullet_{\mathrm{cts}}(G, A) = \cdots \xrightarrow{\partial^{n-1}_{\mathrm{cts}}} X^{n-1}_{\mathrm{cts}}(G, A) \xrightarrow{\partial^n_{\mathrm{cts}}} X^n_{\mathrm{cts}}(G, A) \xrightarrow{\partial^{n+1}_{\mathrm{cts}}} \cdots .$$

As usual, we then set

$$C^n_{\mathrm{cts}}(G, A) \coloneqq X^n_{\mathrm{cts}}(G, A)^G.$$

One checks that $\partial^n_{\mathrm{cts}}$ *restricts to a homomorphism* $C^{n-1}_{\mathrm{cts}}(G, A) \to C^n_{\mathrm{cts}}(G, A)$. *We then let* $C^\bullet_{\mathrm{cts}}(G, A)$ *be the complex*

$$C^\bullet_{\mathrm{cts}}(G, A) = \cdots \xrightarrow{\partial^{n-1}_{\mathrm{cts}}} C^{n-1}_{\mathrm{cts}}(G, A) \xrightarrow{\partial^n_{\mathrm{cts}}} C^n_{\mathrm{cts}}(G, A) \xrightarrow{\partial^{n+1}_{\mathrm{cts}}} \cdots .$$

This complex is called the **continuous standard resolution** *of G with coefficients in A. We denote its n-th cohomology group by* $H^n_{\mathrm{cts}}(G, A)$ *and call it the n-th* **continuous cohomology group** *of G with coefficients in A.*

Proposition 2.1.9.
Let G be a profinite group and A be an abelian topological group on which G acts continuously. Then $H^n_{\mathrm{cts}}(G, \mathrm{Map}_{\mathrm{cts}}(G, A)) = 0$ *for all $n > 0$.*

Proof.
The proof for the discrete case is at [NSW15, (1.3.7) Proposition, p. 32]. By proving that the involved maps are well defined, this proof transforms to our situation. In particular, we will show that the maps

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{cts}}(G^{n+1}, \mathrm{Map}_{\mathrm{cts}}(G, A))^G & \longleftrightarrow & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A) \\
x & \xmapsto{\ \alpha\ } & [(\sigma_0, \dots, \sigma_n) \mapsto x(\sigma_0, \dots, \sigma_n)(1)] \\
{}[(\sigma_0, \dots, \sigma_n) \mapsto [\sigma \mapsto \sigma(y(\sigma^{-1}\sigma_0 \dots \sigma^{-1}\sigma_n))]] & \xleftarrow[\ \beta\ ]{} & y
\end{array}$$

are well defined and inverse to each other. For this, we will write $\alpha_x \coloneqq \alpha(x)$ and $\beta_y \coloneqq \beta(y)$.
For the well definedness we have to show that the maps α_x, β_y and $\beta_y(\sigma_0, \dots, \sigma_n)$ are continuous for all $x \in \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, \mathrm{Map}_{\mathrm{cts}}(G, A))$, $y \in \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$ and $(\sigma_0 \dots, \sigma_n) \in G^{n+1}$ as well as β_y is fixed by the operation of G.

So, let $x \in \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, \mathrm{Map}_{\mathrm{cts}}(G,A))$, $U \subseteq A$ open and $(\sigma_0, \dots, \sigma_n) \in G^{n+1}$ such that $x(\sigma_0, \dots, \sigma_n)(1) \in U$, i.e. $(\sigma_0, \dots, \sigma_n) \in \alpha_x^{-1}(U)$. We then clearly have $x(\sigma_0, \dots, \sigma_n) \in \mathrm{M}(\{1\}, U)$. Since x is continuous, there exists an open $V \subseteq G^{n+1}$ with $(\sigma_0, \dots, \sigma_n) \in V$ such that $V \subseteq x^{-1}(\mathrm{M}(\{1\}, U))$. But then we also have $V \subseteq \alpha_x^{-1}(U)$, which prooves the continuity of α_x.
Now let $y \in \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$. Theorem 2.1.1 says that β_y is continuous if the map

$$G \times G^{n+1} \longrightarrow A,\ (\sigma, (\sigma_0, \dots, \sigma_n)) \longmapsto \sigma(y(\sigma^{-1}\sigma_0, \dots, \sigma^{-1}\sigma_n))$$

is continuous. This map can be written as the composite of the following maps

$$\begin{array}{ccccccc}
G \times G^{n+1} & \longrightarrow & G \times G^{n+1} & \longrightarrow & G \times A & \longrightarrow & A \\
(\sigma, \tau) & \longmapsto & (\sigma, \sigma^{-1} \cdot \tau) & & & & \\
& & (\sigma, \tau) & \longmapsto & (\sigma, y(\tau)) & & \\
& & & & (\sigma, a) & \longmapsto & \sigma(a).
\end{array}$$

The first of these maps is continuous because inversion in G is continuous and multiplication in G is continuous, therefore the componentwise action of G on G^{n+1} by multiplication is also continuous. The second map is continuous since y is and the last map is continuous since G acts continuously on A. In conclusion β_y is continuous.
Next we show that β_y is fixed under the operation from G. Let for this $\eta, \sigma \in G$ and $\tau \in G^{n+1}$. Then we have:

$$\begin{aligned}
((\eta \cdot \beta_y)(\tau))(\sigma) &= \left(\eta \cdot (\beta_y(\eta^{-1}\tau))\right)(\sigma) \\
&= \eta(\beta_y(\eta^{-1}\tau)(\eta^{-1}\sigma)) \\
&= \eta((\eta^{-1}\sigma)y(\sigma^{-1}\eta\eta^{-1}\tau)) \\
&= \sigma(y(\sigma^{-1}\tau)) \\
&= (\beta_y(\tau))(\sigma),
\end{aligned}$$

i.e. β_y is fixed under the operation of G.
Let now additionally $\tau := (\sigma_0, \dots, \sigma_n) \in G^{n+1}$ and $U \subseteq A$ be open. Let $\sigma \in G$ such that $\beta_y(\tau)(\sigma) = \sigma(y(\sigma^{-1} \cdot \tau)) \in U$. First note that $\beta_y(\tau)(\sigma) = (\sigma \cdot y)(\tau)$ and that $\sigma \cdot y$ again is continuous (cf. proof of Proposition 2.1.5). Then, since $\sigma \cdot y$ is continuous and G^{n+1} is compact (since G is a profinite group) and therefore also locally compact, it exists a compact neighboruhood $K \subseteq G^{n+1}$ of τ such that $(\sigma \cdot y)(K) \subseteq U$ (cf. [Bou89a, Corollary to Proposition 9, Chapter I §9.7, p. 90]), i.e. $\sigma \cdot y \in \mathrm{M}(K, U)$. Since G acts continuously on $\mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$ (cf. Proposition 2.1.5) then exist open sets $V \subseteq G$ and $W \subseteq \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$ such that $\sigma \in V$, $y \in W$ and $V \times W \subseteq \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)$.

Especially we have $\eta \cdot y \in \mathrm{M}(K, U)$ for all $\eta \in V$ and since $\tau \in K$ we then get $\beta_y(\tau)(\eta) = (\eta \cdot y)(\tau) \in U$ for all $\eta \in V$, i.e. V is an open neighbourhood of σ contained in $\beta_y(\tau)^{-1}(U)$, so $\beta_y(\tau)$ is continuous.
The rest of the proof now follows [NSW15, (1.3.7) Proposition, p. 32], but here we use continuous cohomology. We actually proved that for every $n > 0$ we have an isomorphism of groups

$$\mathrm{Map}_{\mathrm{cts}}(G^n, \mathrm{Map}_{\mathrm{cts}}(G, A))^G \cong X_{\mathrm{cts}}(G^n, A).$$

We want this to be an isomorphism of complexes, so we have to check that it commutes with the corresponding differentials. Thus, we have to check that for all $n > 0$ the following diagrams commute:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{cts}}(G^n, \mathrm{Map}_{\mathrm{cts}}(G, A))^G \arrow[r, "\alpha"] \arrow[d, "\partial^n_{\mathrm{cts}}"'] & \mathrm{Map}_{\mathrm{cts}}(G^n, A) \arrow[d, "\partial^n_{\mathrm{cts}}"] \\
\mathrm{Map}_{\mathrm{cts}}(G^{n+1}, \mathrm{Map}_{\mathrm{cts}}(G, A))^G \arrow[r, "\alpha"] & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{cts}}(G^n, A) \arrow[r, "\beta"] \arrow[d, "\partial^n_{\mathrm{cts}}"'] & \mathrm{Map}_{\mathrm{cts}}(G^n, \mathrm{Map}_{\mathrm{cts}}(G, A))^G \arrow[d, "\partial^n_{\mathrm{cts}}"] \\
\mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A) \arrow[r, "\beta"] & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, \mathrm{Map}_{\mathrm{cts}}(G, A))^G.
\end{tikzcd}$$

So, let $x \in \mathrm{Map}_{\mathrm{cts}}(G^n, \mathrm{Map}_{\mathrm{cts}}(G, A))^G$. Then it is

$$(\alpha \circ \partial^n_{\mathrm{cts}})(x)(\sigma_0, \dots, \sigma_n) = \sum_{i=0}^{n} (x(\sigma_0, \dots, \widehat{\sigma_i}, \dots, \sigma_n))(1).$$

On the other hand, we have

$$\alpha(x)(\sigma_0, \dots, \sigma_{n-1}) = x(\sigma_0, \dots, \sigma_{n-1})(1)$$

and therefore

$$(\partial^n_{\mathrm{cts}} \circ \alpha)(x)(\sigma_0, \dots, \sigma_n) = \sum_{i=0}^{n} \alpha(x)(\sigma_0, \dots, \widehat{\sigma_i}, \dots, \sigma_n) = \sum_{i=0}^{n} x(\sigma_0, \dots, \widehat{\sigma_i}, \dots, \sigma_n)(1),$$

i.e. the first diagram commutes. For the second diagram let $y \in \mathrm{Map}_{\mathrm{cts}}(G^n, A)$. Then

we have

$$\begin{aligned}(\partial^n_{\mathrm{cts}} \circ \beta)(y)(\sigma_0, \ldots, \sigma_n)(\sigma) &= \sum_{i=0}^{n} \beta(y)(\sigma_0, \ldots, \widehat{\sigma_i}, \ldots, \sigma_n)(\sigma) \\ &= \sum_{i=0}^{n} \sigma(y(\sigma^{-1}\sigma_0, \ldots, \widehat{\sigma^{-1}\sigma_i}, \ldots, \sigma^{-1}\sigma_n)).\end{aligned}$$

On the other hand we have

$$\begin{aligned}((\beta \circ \partial^n_{\mathrm{cts}})(y)(\sigma_0, \ldots, \sigma_n))(\sigma) &= \sigma(\delta^n(y)(\sigma^{-1}\sigma_0, \ldots, \sigma^{-1}\sigma_n)) \\ &= \sigma\left(\sum_{i=0}^{n} y(\sigma^{-1}\sigma_0, \ldots, \widehat{\sigma^{-1}\sigma_i}, \ldots, \sigma^{-1}\sigma_n)\right),\end{aligned}$$

i.e. the second diagram commutes. Thus, we have an isomorphism of complexes

$$C^{\bullet}_{\mathrm{cts}}(G, \mathrm{Map}_{\mathrm{cts}}(G, A)) \cong X^{\bullet}_{\mathrm{cts}}(G, A).$$

The complex $X^{\bullet}_{\mathrm{cts}}(G, A)$ is exact (cf. Lemma 2.1.7) and therefore we have

$$H^n_{\mathrm{cts}}(G, \mathrm{Map}_{\mathrm{cts}}(G, A)) = H^n(C^{\bullet}_{\mathrm{cts}}(G, \mathrm{Map}_{\mathrm{cts}}(G, A))) \cong H^n(X^{\bullet}_{\mathrm{cts}}(G, A)) = 0.$$

□

Lemma 2.1.10.
Let G be a profinite group and A a G-module. Then for all $n > 0$ the G-module $\mathrm{Map}_{\mathrm{cts}}(G^n, A)$ is G-induced.

Proof.
Since G is Hausdorff and compact Proposition 2.1.3 says that the canonical maps

$$\begin{array}{rcl}\mathrm{Map}_{\mathrm{cts}}(G^n, A) & \xrightarrow{\cong} & \mathrm{Map}_{\mathrm{cts}}(G, \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, A)), \\ f & \xmapsto{\alpha} & [\sigma \mapsto [(\sigma_1, \ldots, \sigma_{n-1}) \mapsto f(\sigma, \sigma_1, \ldots, \sigma_{n-1})]] \\ {[(\sigma_1, \ldots, \sigma_n) \mapsto f(\sigma_1)(\sigma_2, \ldots, \sigma_n)]} & \xleftarrow[\beta]{} & f\end{array}$$

are homeomorphisms. These maps are also compatible with the group structure on both sides. To see this for α let $f, g \in \mathrm{Map}_{\mathrm{cts}}(G^n, A)$ and $\sigma, \sigma_1, \ldots, \sigma_{n-1} \in G$ and compute:

$$\begin{aligned}(\alpha(f+g)(\sigma))(\sigma_1,\dots,\sigma_{n-1}) &= (f+g)(\sigma,\sigma_1,\dots,\sigma_{n-1})\\ &= f(\sigma,\sigma_1,\dots,\sigma_{n-1})+g(\sigma,\sigma_1,\dots,\sigma_{n-1})\\ &= (\alpha(f(\sigma))+\alpha(g(\sigma)))(\sigma_1,\dots,\sigma_{n-1})\\ &= ((\alpha(f)+\alpha(g))(\sigma))(\sigma_1,\dots,\sigma_{n-1}).\end{aligned}$$

For β let $f,g \in \mathrm{Map}_{\mathrm{cts}}(G,\mathrm{Map}_{\mathrm{cts}}(G^{n-1},A))$ and $\sigma_1,\dots,\sigma_n \in G$ and compute:

$$\begin{aligned}\beta(f+g)(\sigma_1,\dots,\sigma_n) &= (f+g)(\sigma_1)(\sigma_2,\dots,\sigma_n)\\ &= (f(\sigma_1)+g(\sigma_1))(\sigma_2,\dots,\sigma_n)\\ &= f(\sigma_1)(\sigma_2,\dots,\sigma_n)+g(\sigma_1)(\sigma_2,\dots,\sigma_n)\\ &= \beta(f)(\sigma_1,\dots,\sigma_n)+\beta(g)(\sigma_1,\dots,\sigma_n)\\ &= (\beta(f)+\beta(g))(\sigma_1,\dots,\sigma_n).\end{aligned}$$

Last we have to see that these maps are compatible with the operation of G. Let $\tau,\sigma,\sigma_1,\dots,\sigma_n \in G$ and $f \in \mathrm{Map}_{\mathrm{cts}}(G^n,A)$. Then we compute

$$\begin{aligned}((\alpha(\tau\cdot f))(\sigma))(\sigma_1,\dots,\sigma_{n-1}) &= (\tau\cdot f)(\sigma,\sigma_1,\dots,\sigma_{n-1})\\ &= \tau(f(\tau^{-1}\sigma,\tau^{-1}\sigma_1,\dots,\tau^{-1}\sigma_{n-1}))\\ &= \tau((\alpha(f)(\tau^{-1}\sigma))(\tau^{-1}\sigma_1,\dots,\tau^{-1}\sigma_{n-1}))\\ &= (\tau\cdot\alpha(f)(\tau^{-1}\sigma))(\sigma_1,\dots,\sigma_{n-1})\\ &= ((\tau\cdot\alpha(f))(\sigma))(\sigma_1,\dots,\sigma_{n-1}),\end{aligned}$$

i.e. it is $\alpha(\tau\cdot f)=\tau\cdot\alpha(f)$. Let now $f \in \mathrm{Map}_{\mathrm{cts}}(G,\mathrm{Map}_{\mathrm{cts}}(G^{n-1},A))$ and compute

$$\begin{aligned}\beta(\tau\cdot f)(\sigma_1,\dots,\sigma_n) &= (\tau\cdot f)(\sigma_1)(\sigma_2,\dots,\sigma_n)\\ &= ((\tau\cdot f)(\tau^{-1}\sigma_1))(\sigma_2,\dots,\sigma_n)\\ &= \tau(f(\tau^{-1}\sigma_1)(\tau^{-1}\sigma_2,\dots,\tau^{-1}\sigma_n))\\ &= \tau(\beta(f)(\tau^{-1}\sigma_1,\dots,\tau^{-1}\sigma_n))\\ &= (\tau\cdot\beta(f))(\sigma_1,\dots,\sigma_n),\end{aligned}$$

i.e. it is $\beta(\tau\cdot f)=\tau\cdot\beta(f)$. In conclusion we have shown that $\mathrm{Map}_{\mathrm{cts}}(G^n,A)$ and $\mathrm{Map}_{\mathrm{cts}}(G,\mathrm{Map}_{\mathrm{cts}}(G^{n-1},A))$ are isomorphic as topological G-modules and therefore $\mathrm{Map}_{\mathrm{cts}}(G^n,A)$ is G-induced. $\square$

Lemma 2.1.11.
Let G be a profinite group and A a G-module. Then for all $n > 0$ we have an isomorphism

$$\begin{aligned} \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)^G &\longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, A) \\ x &\longmapsto [(\sigma_1, \dots, \sigma_n) \mapsto x(\textstyle\prod_{i=1}^{j} \sigma_i)_{j=0}^{n}] \\ [(\sigma_0, \dots, \sigma_n) \mapsto \sigma_0 y(\sigma_{i-1}^{-1}\sigma_i)_{i=1}^{n}] &\longleftarrow\!\shortmid\; y \end{aligned}$$

of abelian groups.

Proof.
For discrete coefficients, this is stated at [NSW15, p. 14]. Since there is no proper reference, we check that the homomorphisms are inverse to each other. That they are well defined is obvious. Denote for this proof the upper homomorphism by f, the image of $x \in \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)^G$ by f_x, the lower homomorphism by g and the image of $y \in \mathrm{Map}_{\mathrm{cts}}(G^n, A)$ by g_y. Let furthermore $\sigma_0, \dots, \sigma_n \in G$. We then compute

$$\begin{aligned} g_{f_x}(\sigma_0, \dots, \sigma_n) &= \sigma_0 f_x(\sigma_0^{-1}\sigma_1, \dots, \sigma_{n-1}^{-1}\sigma_n) \\ &= \sigma_0 x(1, \sigma_0^{-1}\sigma_1, \sigma_0^{-1}\sigma_2, \dots, \sigma_0^{-1}\sigma_n) \\ &= x(\sigma_0, \dots, \sigma_n) \end{aligned}$$

where the last equation is true since x is fixed under the operation of G. For the other direction we compute

$$\begin{aligned} f_{g_y}(\sigma_1, \dots, \sigma_n) &= g_y(1, \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1 \cdots \sigma_n) \\ &= y(\sigma_1, \dots, \sigma_n). \end{aligned}$$

So f and g are inverse to each other. $\square$

Corollary 2.1.12.
Let G be a profinite group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of topological G-modules such that the topology of A is induced by that of B and that $B \to C$ has a continuous set theoretical section $s\colon C \to B$.

Then for all $n > 0$ the diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, A) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, A) \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, B) \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, C) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, A)^G \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, B)^G \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{cts}}(G^{n}, C)^G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A)^G \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, B)^G \arrow[r] & \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, C)^G \arrow[r] & 0
\end{tikzcd}$$

are commutative with exact rows and the latter diagram induces a long exact sequence of continuous cohomology

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1_{\mathrm{cts}}(G, A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^n_{\mathrm{cts}}(G, A) \longrightarrow H^n_{\mathrm{cts}}(G, B) \longrightarrow H^n_{\mathrm{cts}}(G, C) \longrightarrow H^{n+1}_{\mathrm{cts}}(G, A) \longrightarrow \cdots$$

Furthermore, the topology of $\mathrm{Map}_{\mathrm{cts}}(G^n, A)$ *is induced by the topology of* $\mathrm{Map}_{\mathrm{cts}}(G^n, B)$ *and the section* $s\colon C \to B$ *induces a continuous, set theoretical section* $s_*\colon \mathrm{Map}_{\mathrm{cts}}(G^n, C) \to \mathrm{Map}_{\mathrm{cts}}(G^n, B)$.

Proof.
First we want to see that the topology of $\mathrm{Map}_{\mathrm{cts}}(G^n, A)$ is induced from the topology of $\mathrm{Map}_{\mathrm{cts}}(G^n, B)$. Let $K \subseteq G^n$ be compact and $U \subseteq A$ be open. Then there exists $V \subseteq B$ open such that $U = V \cap A$. Then we have

$$M(K, U) = M(K, V \cap A) = M(K, V) \cap M(K, A) = M(K, V) \cap \mathrm{Map}_{\mathrm{cts}}(G^n, A).$$

Since $\mathrm{Map}_{\mathrm{cts}}(G^n, -)$ turns continuous maps into continuous maps, the map $s_*\colon \mathrm{Map}_{\mathrm{cts}}(G^n, C) \to \mathrm{Map}_{\mathrm{cts}}(G^n, B)$ induced from the continuous section $s\colon C \to B$ again is continuous (with the same argument are α_* and β_* seen to be continuous). s_* is a section as well, because for $f \in \mathrm{Map}_{\mathrm{cts}}(G^n, C)$ we get

$$(\beta_* \circ s_*)(f) = \beta_*(s_*(f)) = \beta \circ s \circ f = f.$$

The commutativity of both diagrams is obvious, so it remains to check that they have exact lines. First, we want to show that the sequence

$$0 \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, A) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, B) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, C) \longrightarrow 0$$

is exact for all $n \geq 0$. For this, we obtain that $\alpha_*(f) = \alpha_*(g)$ for $f, g \in \mathrm{Map}_{\mathrm{cts}}(G^n, A)$ if and only if $\alpha \circ f = \alpha \circ g$, which is equivalent to $f = g$ since α is injective. Furthermore, for $f \in \mathrm{Map}_{\mathrm{cts}}(G^n, A)$ we have

$$(\beta_* \circ \alpha_*)(f) = \beta \circ \alpha \circ f = 0,$$

since $\beta \circ \alpha = 0$, i.e. it is $\mathrm{im}(\alpha_*) \subseteq \ker(\beta_*)$. For the opposite inclusion let $f \in \ker(\beta_*)$. Then it is $\beta(f(x)) = 0$ for all $x \in G^n$, i.e. it is $f(x) \in \mathrm{im}(\alpha)$ for all $x \in G^n$. We then define a map $g\colon G^n \to A$ by $g(x) := \alpha^{-1}(f(x))$. This is well defined and continuous since α is a homeomorphism from A to $\mathrm{im}\,\alpha$ because we assumed that the topology of A is induced by that of B. Last we have to see that β_* is surjective. But for $f \in \mathrm{Map}_{\mathrm{cts}}(G^n, C)$ we may set $g := s_*(f) = s \circ f$ which then is a continuous map from G^n to B with $\beta_*(g) = \beta \circ g = \beta \circ s \circ f = f$.
The exactness for the sequence

$$0 \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, A)^G \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, B)^G \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^n, C)^G \longrightarrow 0$$

for $n \geq 0$ then follows with Lemma 2.1.11 and the exactness of

$$0 \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, A) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, B) \longrightarrow \mathrm{Map}_{\mathrm{cts}}(G^{n-1}, C) \longrightarrow 0.$$

As in [NSW15, Chapter I, §3, (1.3.2) Theorem, p. 27] the long exact sequence of cohomology then is an application of the snake lemma, here in its topological version (cf. [Sch99, Proposition 4, p. 133-134]). $\square$

2.2 Monoid Cohomology

As described in the introduction, the aim of Chapter 4 is to compute Galois cohomology using the theory of Lubin-Tate (φ, Γ)-modules. For this, we also compute the cohomology of complexes like $A \xrightarrow{f-1} A$, where A is a topological abelian group and f is a continuous endomorphism of A.
This can be embedded in the theory of monoid cohomology, which then allows us, in the case of discrete coefficients, to write this cohomological functor as derived functor. We then combine this with a usual group action, which commutes with the endomorphism and obtain spectral sequences on cohomology.

Proposition 2.2.1.
Let A be a topological abelian group and $f \in \operatorname{End}(A)$ continuous. Then

$$\cdot : \mathbb{N}_0 \times A \longrightarrow A, \ (n,a) \longmapsto f^n(a)$$

defines a continuous $\mathbb{N}_0$-action on A.

Proof.
The properties 1.-3. of Definition 2.1.4 are immediately clear. For the continuity let $U \subseteq A$ be open and $(n,a) \in \mathbb{N}_0 \times A$ such that $f^n(a) \in U$. Since f is continuous, f^n is continuous as well and therefore $(f^n)^{-1}(U) \subseteq A$ is an open set. But then $\{n\} \times (f^n)^{-1}(U) \subseteq \mathbb{N}_0 \times A$ is an open neighbourhood of (n,a) contained in the preimage of U under $\cdot$. □

Proposition 2.2.2.
Let M be a topological monoid and A be a discrete abelian group with a continuous action of M. Then we have

$$A^M \cong \operatorname{Hom}_{\mathbb{Z}[M]}(\mathbb{Z}, A),$$

as $\mathbb{Z}[M]$-modules, where $\mathbb{Z}$ is considered as trivial $\mathbb{Z}[M]$-module.

Proof.
We consider the following maps

$$\begin{array}{rcl} A^M & \longleftrightarrow & \operatorname{Hom}_{\mathbb{Z}[M]}(\mathbb{Z}, A) \\ a & \xmapsto{\alpha} & [x \mapsto x \cdot a] \\ f(1) & \xleftarrow[\beta]{} & f. \end{array}$$

These maps are clearly homomorphisms and they are continuous and open, since both, $\mathbb{Z}$ and A, are discrete.

1. α is well defined:
   Let $a \in A^M$ and $m \in M$. With a it also is $x \cdot a \in A^M$ and therefore we have

   $$m \cdot \alpha(a)(x) = m \cdot (x \cdot a) = x \cdot a = \alpha(a)(x) = \alpha(a)(m \cdot x),$$

   i.e. $\alpha(a)$ is $\mathbb{Z}[M]$-linear.

2. β is well defined:
   Let $f \in \operatorname{Hom}_{\mathbb{Z}[M]}(\mathbb{Z}, A)$ and $m \in M$. Since f is $\mathbb{Z}[M]$-linear and $\mathbb{Z}$ is a trivial

M-module, we then get

$$m \cdot f(1) = f(m \cdot 1) = f(1),$$

i.e. $f(1) \in A^M$.

3. $\alpha \circ \beta = \mathrm{id}_{\mathrm{Hom}_{\mathbb{Z}[M]}(\mathbb{Z},A)}$:
   Let $f \in \mathrm{Hom}_{\mathbb{Z}[M]}(\mathbb{Z}, A)$. For $x \in \mathbb{Z}$ we then obtain:

   $$\alpha(\beta(f))(x) = x \cdot \beta(f) = x \cdot f(1) = f(x),$$

   where the last equality is true, since f is $\mathbb{Z}$-linear. This immediately gives $(\alpha \circ \beta)(f) = f$, i.e. $\alpha \circ \beta = \mathrm{id}_{\mathrm{Hom}_{\mathbb{Z}[M]}(\mathbb{Z},A)}$.

4. $\beta \circ \alpha = \mathrm{id}_{A^M}$:
   Let $a \in A^M$. Then we get:

   $$\beta(\alpha(a)) = \alpha(a)(1) = 1 \cdot a = a,$$

   i.e. $\beta \circ \alpha = \mathrm{id}_{A^M}$.

□

We are mostly interested in the case of a discrete G-module A, where G is a profinite group, together with an $\mathbb{N}_0$-action (which then automatically is continuous since both, $\mathbb{N}_0$ and A are discrete), which comes from a G-homomorphism of A. To shorten notation, we make the following definitions.

Definition 2.2.3.
Let G be a profinite group and M a topological monoid.
By $\mathcal{DIS}_M$ we denote the category whose objects are discrete abelian groups with a continuous action of M and whose morphisms are the continuous group homomorphisms which respect the operation of M.
Similarly we denote by $\mathcal{DIS}_G$ the category whose objects are discrete abelian groups with a continuous action of G and whose morphisms are the continuous group homomorphisms which respect the operation of G.
And finally we denote by $\mathcal{DIS}_{G,M}$ the category whose objects are discrete abelian groups, together with commuting continuous actions of G and M and whose morphisms are the continuous group homomorphisms which respect the operations from G and M.
The corresponding categories, whose objects are abstract abelian groups, are denoted

by $\mathcal{ABS}_M$, $\mathcal{ABS}_G$ and $\mathcal{ABS}_{GM}$.
Furthermore, by $\mathcal{TOP}_G$ we denote the category of topological abelian Hausdorff groups with a continuous action from G. The morphisms of this category are the continuous group homomorphisms which respect the action from G.
Analogously we denote by $\mathcal{TOP}_{G,M}$ the category of topological abelian Hausdorff groups with continuous actions from both, G and M, such that these actions commute. The morphisms of this category are the continuous group homomorphisms which respect the actions from G and M.

Remark 2.2.4.
Let G be a profinite group and M a topological monoid. Then the categories $\mathcal{DIS}_{\mathrm{G,M}}$ and $\mathcal{DIS}_{\mathrm{G\times M}}$ coincide, where $G \times M$ is considered as a topological monoid.

Proof.
If $A \in \mathcal{DIS}_{\mathrm{G\times M}}$ then by $g \cdot a := (g, 1) \cdot a$ respectively $m \cdot a := (1, m) \cdot a$ for all $g \in G$, $m \in M$ and $a \in A$ we can define operations from G and M on A which then are automatically continuous, since the action from $G \times M$ on A is continuous. Because of $(g, m) = (g, 1)(1, m) = (1, m)(g, 1)$ for all $g \in G$ and $m \in M$, it is immediately clear that these actions commute. Therefore it is $A \in \mathcal{DIS}_{\mathrm{G,M}}$.
If $A \in \mathcal{DIS}_{\mathrm{G,M}}$, then one can define an action of $G\times M$ on A by $(g, m)\cdot a := g\cdot(m\cdot a)$. Since the actions of G and M commute, this is a well defined $G \times M$-action on A. It is continuous, because it can be factored as the composite of the following maps

$$\begin{array}{rcccl} (G \times M) \times A & \longrightarrow & G \times A & \longrightarrow & A \\ (g, m, a) & \longmapsto & (g, m \cdot a) & & \\ & & (g, a) & \longmapsto & g \cdot a. \end{array}$$

Since both of the above maps are continuous, so is their composite, which is the action from $G \times M$.
That the morphisms coincide is obvious from the definitions of the actions. $\square$

Our aim now is to see that the category $\mathcal{DIS}_{\mathrm{G,M}}$ has enough injective objects. For this, we follow the idea of [NSW15, (2.6.5) Lemma, Chapter I §6, p. 131] and outline some details.

Proposition 2.2.5.
Let G be a group and M a monoid.
Then the category $\mathcal{ABS}_{\mathrm{G,M}}$ coincides with the category of $\mathbb{Z}[G][M]$-modules.

Proof.
The only question which maybe is not immediately clear, is: If we have a $\mathbb{Z}[G][M]$-

module A, why do the operations from G and M commute. But this comes directly from the definition of $\mathbb{Z}[G][M]$. There we havs $g \cdot m = m \cdot g$ for all $g \in G$ and $m \in M$. Therefore we have

$$g \cdot (m \cdot a) = (g \cdot m) \cdot a = (m \cdot g) \cdot a = m \cdot (g \cdot a)$$

for all $g \in G$, $m \in M$ and $a \in A$. □

Corollary 2.2.6.
The category $\mathcal{ABS}_{\mathrm{G,M}}$ has enough injectives.

Proof.
Since the category of R-modules for an arbitrary ring R has enough injectives, this is an immediate consequence from Proposition 2.2.5. □

Lemma 2.2.7.
Let G be a profinite group, M a discrete monoid and $A \in \mathcal{ABS}_{\mathrm{G,M}}$. Define

$$A^\delta := \bigcup_{U \leq G \text{ open}} A^U.$$

Then $A^\delta \in \mathcal{DIS}_{\mathrm{G,M}}$.

Proof.
We endow A^δ with the discrete topology and deduce from [NSW15, (1.1.8) Proposition, Chapter I §1, p. 7–8] that $A^\delta \in \mathcal{DIS}_{\mathrm{G}}$. Now let $a \in A^\delta$ and $m \in M$. Then there exists $U \leq G$ open, such that $a \in A^U$. Since, by definition, the actions of M and G commute, we obtain for all $u \in U$

$$u \cdot (m \cdot a) = (u \cdot m) \cdot a = (m \cdot u) \cdot a = m \cdot (u \cdot a) = m \cdot a,$$

i.e. $m \cdot a \in A^U$ and therefore M also acts in A^δ. Since both, M and A^δ carry the discrete topology, this action trivially is continuous and since the actions of G and M commute on A, their restrictions on A^δ do so as well. This means that we have $A^\delta \in \mathcal{DIS}_{\mathrm{G,M}}$ as claimed. □

Corollary 2.2.8.
Let G be a profinite group, M a discrete monoid and $A \in \mathcal{ABS}_{\mathrm{G,M}}$. Then

$$A^G = (A^\delta)^G.$$

Proof.
Clear, since $A^\delta \subseteq A$ and $A^G \subseteq A^\delta$. □

Lemma 2.2.9.
Let G be a profinite group, $A \in \mathcal{DIS}_{\mathrm{G}}$ and $B \in \mathcal{ABS}_{\mathrm{G}}$. Let furthermore $f\colon A \to B$ be a group homomorphism which respects the actions of G. Then $\mathrm{im}(f) \subseteq B^\delta$, i.e. $f\colon A \to B^\delta$ is a morphism in $\mathcal{DIS}_{\mathrm{G}}$.

Proof.
Let $y \in \mathrm{im}(f)$ and $x \in A$ such that $f(x) = y$. Since $A \in \mathcal{DIS}_{\mathrm{G}}$ it is $A = A^\delta$ (cf. [NSW15, (1.1.8) Proposition, Chapter I §1, p. 7–8]) and therefore it exists $U \leq G$ open such that $x \in A^U$. For all $u \in U$ we then deduce

$$u \cdot y = u \cdot f(x) = f(u \cdot x) = f(x) = y,$$

i.e. $y \in B^U \subseteq B^\delta$. □

Corollary 2.2.10.
Let G be a profinite group, M a discrete monoid, $A \in \mathcal{DIS}_{\mathrm{G,M}}$ and $B \in \mathcal{ABS}_{\mathrm{G,M}}$. If $f\colon A \to B$ is a group homomorphism which respects the actions from G and M then $\mathrm{im}(f) \subseteq B^\delta$ and $f\colon A \to B^\delta$ is a morphism in $\mathcal{DIS}_{\mathrm{G,M}}$.

Proof.
The first part is an immediate consequence of Lemma 2.2.9, the second is direct from the assumption: If $f\colon A \to B$ respects the operations from G and M then $f\colon A \to \mathrm{im}(f)$ does so as well and with $\mathrm{im}(f) \subseteq B^\delta$ as well as $B^\delta \in \mathcal{DIS}_{\mathrm{G,M}}$ (cf. Lemma 2.2.7) the claim follows. □

Lemma 2.2.11.
Let G be a profinite group, M a discrete monoid and $I \in \mathcal{ABS}_{\mathrm{G,M}}$ an injective object. Then $I^\delta \in \mathcal{DIS}_{\mathrm{G,M}}$ also is an injective object.

Proof.
Let $A, B \in \mathcal{DIS}_{\mathrm{G,M}}$, $f\colon A \to B$ injective and $u\colon A \to I^\delta$ and consider the following

diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \\ & I^{\delta} \arrow[d] & \\ & I. & \end{tikzcd}$$

Since A and B are also objects in $\mathcal{ABS}_{\mathrm{G,M}}$ and I is injective, there exists a morphism $v\colon B \to I$ in $\mathcal{ABS}_{\mathrm{G,M}}$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[ddl, "v"] \\ & I^{\delta} \arrow[d] & \\ & I. & \end{tikzcd}$$

From Corollary 2.2.10 we deduce that $\mathrm{im}(v) \subseteq I^{\delta}$ and that $v\colon B \to I^{\delta}$ is a morphism in $\mathcal{DIS}_{\mathrm{G,M}}$. This morphism still fulfils $u = v \circ f$, which then means that $I^{\delta} \in \mathcal{DIS}_{\mathrm{G,M}}$ is an injective object. $\square$

Proposition 2.2.12.
Let G be a profinite group and M a discrete monoid. Then the category $\mathcal{DIS}_{\mathrm{G,M}}$ has enough injective objects.

Proof.
Let $A \in \mathcal{DIS}_{\mathrm{G,M}}$. Since also $A \in \mathcal{ABS}_{\mathrm{G,M}}$ and $\mathcal{ABS}_{\mathrm{G,M}}$ has enough injective objects (cf. Corollary 2.2.6) we can find an injective object $I \in \mathcal{ABS}_{\mathrm{G,M}}$ together with an inclusion $A \to I$ in $\mathcal{ABS}_{\mathrm{G,M}}$. From Corollary 2.2.10 we then deduce an inclusion $A \to I^{\delta}$ in $\mathcal{DIS}_{\mathrm{G,M}}$ and from Lemma 2.2.11 that I^{δ} is an injective object in $\mathcal{DIS}_{\mathrm{G,M}}$, which ends the proof. $\square$

Lemma 2.2.13.
Let G be a profinite group and M a discrete monoid. Then the functor

$$(-)^{G,M}\colon \mathcal{DIS}_{\mathrm{G,M}} \to \mathbf{Ab}$$

is left exact and additive ($\mathbf{Ab}$ denotes the category of abelian groups).

Proof.

Since $\mathcal{DIS}_{\mathrm{G,M}}$ and $\mathcal{DIS}_{\mathrm{G\times M}}$ coincide (cf. Remark 2.2.4) we can view the functor $(-)^{G,M}$ as $(-)^{G\times M}$. Then Proposition 2.2.2 says

$$(-)^{G\times M} = \mathrm{Hom}_{\mathbb{Z}[G\times M]}(\mathbb{Z}, -)$$

which immediately gives the claim, since $\mathrm{Hom}(\mathbb{Z}, -)$ is left exact and additive. □

Proposition 2.2.12 and Lemma 2.2.13 together say that the right derivations for $(-)^{G,M}$, where G is a profinite group and M a discrete monoid, exist (cf. [Sta18, Tag 0156, Lemma 10.3.2 (2)]). This then leads us to the following definition.

Definition 2.2.14.
Let G be a profinite group and M a discrete monoid. Then $H^n(G, M; -) := R^n(-)^{G,M}$ denotes the n-th right derived functor of $(-)^{G,M}$ and is called the **n-th cohomology group**.

Remark 2.2.15.
Recall that the right derived functors are computed by choosing an injective resolution, i.e. if G is a profinite group, M a discrete monoid and $A \in \mathcal{DIS}_{\mathrm{G,M}}$ and $I^n \in \mathcal{DIS}_{\mathrm{G,M}}$ are injective objects for $n \in \mathbb{N}_0$ such that the complex

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is exact, then it is $H^n(G, M; A) = H^n((I^\bullet)^{G,M})$. Note, that if A itself is an injective object, than it is $H^n(G, M; A) = 0$ for $n > 0$ since then

$$0 \longrightarrow A \longrightarrow A \longrightarrow 0$$

is an injective resolution.

Lemma 2.2.16.
Let G be a profinite group, $N \lhd G$ a closed, normal subgroup and M a discrete monoid. Then the functors

$$\begin{aligned} (-)^{N,M}: \quad & \mathcal{DIS}_{\mathrm{G,M}} \longrightarrow \mathcal{DIS}_{\mathrm{G/N}} \\ (-)^{N}: \quad & \mathcal{DIS}_{\mathrm{G,M}} \longrightarrow \mathcal{DIS}_{\mathrm{G/N,M}} \end{aligned}$$

send injectives to injectives.

Proof.
Let $I \in \mathcal{DIS}_{\mathrm{G,M}}$ an injective object, $A, B \in \mathcal{DIS}_{\mathrm{G/N}}$, $f\colon A \to B$ injective,

$u\colon A \to I^{N,M}$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \\
& I^{N,M} \arrow[d] & \\
& I. &
\end{tikzcd}$$

We then let M trivially act on A and B and define $g \cdot x := [g] \cdot x$ for $g \in G$ and $x \in A$ respectively $x \in B$. Here $[g]$ denotes the class of g in G/N. This then defines a continuous G action on both, A and B. Since this action obviously commutes with the trivial action from M we have $A, B \in \mathcal{DIS}_{\mathrm{G,M}}$. Since I is an injective object, we then get a morphism $v\colon B \to I$ in $\mathcal{DIS}_{\mathrm{G,M}}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[ddl, "v"] \\
& I^{N,M} \arrow[d] & \\
& I. &
\end{tikzcd}$$

Let $b \in B$ and $z \in N$ or $z \in M$. Then we have

$$z \cdot f(b) = f(z \cdot b) = f(b),$$

i.e. $\mathrm{im}(f) \subseteq I^{N,M}$ and $v\colon B \to I^{N,M}$ is a morphism in $\mathcal{DIS}_{\mathrm{G/N}}$ which then proves that $I^{N,M}$ is an injective object in $\mathcal{DIS}_{\mathrm{G/N}}$.
In exact the same way, one proves that $I^N \in \mathcal{DIS}_{\mathrm{G/N,M}}$ is an injective object: For $A, B \in \mathcal{DIS}_{\mathrm{G/N,M}}$ one defines a G-action as above and one carries the M-action instead of letting M act trivially. The rest of the proof is literally equal. □

Proposition 2.2.17.
Let G be a profinite group, $N \lhd G$ a closed, normal subgroup and M a discrete monoid. Then for every $A \in \mathcal{DIS}_{\mathrm{G,M}}$ there are two cohomological spectral sequences converging to $H^n(G, M; A)$:

$$H^a(G/N, H^b(N, M; A)) \Longrightarrow H^{a+b}(G, M; A)$$
$$H^a(G/N, M; H^b(N, A)) \Longrightarrow H^{a+b}(G, M; A).$$

Proof.
Proposition 2.2.12 says that the categories $\mathcal{DIS}_{\mathrm{G,M}}$, $\mathcal{DIS}_{\mathrm{G/N,M}}$ and $\mathcal{DIS}_{\mathrm{G/N}}$ have enough injectives. Lemma 2.2.16 says that the functors $(-)^{N,M}\colon \mathcal{DIS}_{\mathrm{G,M}} \to \mathcal{DIS}_{\mathrm{G/N}}$ respectively
$(-)^{N}\colon \mathcal{DIS}_{\mathrm{G,M}} \to \mathcal{DIS}_{\mathrm{G/N,M}}$ send injectives to injectives. Furthermore, since the actions of G and M on objects of $\mathcal{DIS}_{\mathrm{G,M}}$ commute, the compositions

$$\mathcal{DIS}_{\mathrm{G,M}} \xrightarrow{(-)^{N,M}} \mathcal{DIS}_{\mathrm{G/N}} \xrightarrow{(-)^{G/N}} \mathbf{Ab}$$

and

$$\mathcal{DIS}_{\mathrm{G,M}} \xrightarrow{(-)^{N}} \mathcal{DIS}_{\mathrm{G/N,M}} \xrightarrow{(-)^{G/N,M}} \mathbf{Ab}$$

both coincide with $(-)^{G,M}$. This then leads to the claimed Grothendieck spectral sequences. $\square$

As we now have accomplished the abstract theory for our goals, we want to discuss how to compute these cohomology groups when the monoid action arises from an endomorphism. First of all, we want to compare $\mathbb{N}_0$-actions with $\mathbb{Z}[X]$-modules.

Remark 2.2.18.
The category $\mathcal{ABS}_{\mathbb{N}_0}$ coincides with the category of $\mathbb{Z}[X]$-modules.

Proof.
To avoid confusion, we denote the action of $\mathbb{N}_0$ on an abstract abelian group for this proof by "$*$" and the canonical action of $\mathbb{Z}$ by "$\cdot$".
Let $A \in \mathcal{ABS}_{\mathbb{N}_0}$. By $X \cdot a := 1 * a$ we make A into a $\mathbb{Z}[X]$-module. Conversely, if A is a $\mathbb{Z}[X]$-module, then by $n * a := X^n \cdot a$ we get $A \in \mathcal{ABS}_{\mathbb{N}_0}$. With these definitions it is immediately clear, that also the morphisms coincide. $\square$

We made this remark, because we think it's better to think of a $\mathbb{Z}[X]$-module than of an object of $\mathcal{ABS}_{\mathbb{N}_0}$ - just for avoiding confusion. In the following, we will switch between these two concepts without mentioning it.

Remark 2.2.19.
Let G be a profinite group, $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. For every $n \in \mathbb{N}_0$ we can define an $\mathbb{N}_0$-action on $C^n_{\mathrm{cts}}(G, A)$ by operating on the coefficients:

$$(X \cdot f)(\sigma) := X \cdot (f(\sigma)).$$

Remark 2.2.20.
Let $A^{\bullet,\bullet}$ b a (commutative) double complex of abelian groups. We write $\mathrm{Tot}(A^{\bullet,\bullet})$ *for its total complex, by which we mean the complex with objects*

$$\mathrm{Tot}^n(A^{\bullet,\bullet}) \coloneqq \bigoplus_{i+j=n} A^{i,j}$$

and differentials

$$\mathrm{d}^n_{\mathrm{Tot}(A^{\bullet,\bullet})} \coloneqq \bigoplus_{i+j=n} \mathrm{d}^{i,j}_{\mathrm{hor}} \circ \mathrm{pr}_{i-1,j} \oplus (-1)^i \mathrm{d}^{i,j}_{\mathrm{vert}} \circ \mathrm{pr}_{i,j-1}.$$

If $f^{\bullet,\bullet}\colon A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is a morphism of (commutative) double complexes, then

$$\begin{aligned} \mathrm{Tot}^n(f^{\bullet,\bullet})\colon\ \mathrm{Tot}^n(A^{\bullet,\bullet}) &\longrightarrow \mathrm{Tot}^n(B^{\bullet,\bullet}) \\ (a_{ij})_{i+j=n} &\longmapsto (f_{ij}(a_{ij}))_{i+j=n} \end{aligned}$$

defines a morphism of the corresponding total complexes.
If $X^\bullet$ and $Y^\bullet$ are complexes of abelian groups and $g^\bullet\colon X^\bullet \to Y^\bullet$ is a morphism of complexes, then it also is a double complex concentrated in degrees 0 *and* 1 *and we again write* $\mathrm{Tot}(g^\bullet\colon X^\bullet \to Y^\bullet)$ *for its total complex.*

Remark 2.2.21.
Let G be a profinite group and $A \in \mathcal{DIS}_{\mathrm{G}}$. As in [NSW15, p. 12–13] *we omit the subscript "*cts*" for the notations introduced in* Remark 2.1.8*, i.e. we write*

$$X^n(G, A) \coloneqq \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A),$$

∂^n for the differential $X^{n-1}(G, A) \to X^n(G, A)$ and

$$C^n(G, A) \coloneqq X^n(G, A)^G.$$

Definition 2.2.22.
Let G be a profinite group and $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. Then define

$$\begin{aligned} \mathcal{C}^\bullet_X(G, A) &\coloneqq \mathrm{Tot}(C^\bullet(G, A) \xrightarrow{X-1} C^\bullet(G, A)), \\ \mathcal{H}^*_X(G, A) &\coloneqq H^*(\mathcal{C}^\bullet_X(G, A)). \end{aligned}$$

If the $\mathbb{N}_0$-action on A comes from an endomorphism $f \in \mathrm{End}_G(A)$ (cf. Proposition

2.2.1), then we also write

$$\mathcal{C}_f^\bullet(G,A) \coloneqq \mathrm{Tot}(C^\bullet(G,A) \xrightarrow{\mathcal{C}^\bullet(G,f)-\mathrm{id}} C^\bullet(G,A)),$$
$$\mathcal{H}_f^*(G,A) \coloneqq H^*(\mathcal{C}_f^\bullet(G,A)).$$

If $A \in \mathcal{ABS}_{\mathbb{N}_0}$ then we also write $\mathcal{H}_X^*(A)$ for the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in the degrees 0 and 1.

The aim now is to see that the cohomology of the complex $\mathcal{C}_X^\bullet(G,A)$ coincides with the right derived functors of $(-)^{G,\mathbb{N}_0}$. Before proving this, we want to make a smaller step and explain first how to compute the right derived functors of $(-)^{\mathbb{N}_0}$ and that these coincide with the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in degrees 0 and 1.

Proposition 2.2.23.
Let $A \in \mathcal{ABS}_{\mathbb{N}_0}$. Then we have

$$\begin{aligned} H^0(\mathbb{N}_0;A) &= A^{\mathbb{N}_0},\\ H^1(\mathbb{N}_0;A) &= A_{\mathbb{N}_0},\\ H^i(\mathbb{N}_0;A) &= 0 \textit{ for all } i \in \mathbb{Z}\setminus\{0,1\}.\end{aligned}$$

In particular, the right derived functors of $(-)^{\mathbb{N}_0}$ coincide with the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in degrees 0 and 1. Using the notation from above, this means that for all $i \in \mathbb{Z}$ there are natural isomorphisms

$$H^i(\mathbb{N}_0;A) = \mathcal{H}_X^i(A).$$

Proof.
In Proposition 2.2.2 we identified the functors $(-)^{\mathbb{N}_0}$ and $\mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z},-)$. To compute the right derived functors of $\mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z},-)$ for A, we also can compute the right derived functors of $\mathrm{Hom}_{\mathbb{Z}[X]}(-,A)$ for $\mathbb{Z}$. To do this, we need a projective resolution of $\mathbb{Z}$ as $\mathbb{Z}[X]$-module, where X acts as 1. Trivially $\mathbb{Z}[X]$ is a projective $\mathbb{Z}[X]$-module and therefore we get a projective resolution of $\mathbb{Z}$ by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[X] & \longrightarrow & \mathbb{Z}[X] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0\\ & & P(X) & \longmapsto & (X-1)P(X) & & & & \\ & & & & P(X) & \longmapsto & P(1). & & \end{array}$$

This sequence is exact:

The first map is injective since $\mathbb{Z}[X]$ is an integral domain and therefore $(X-1)P(X)$ is zero if and only if $P(X)$ is zero. The second map is surjective, since for $z \in \mathbb{Z}$ the constant polynomial $P_z(X) \coloneqq z$ maps to z. The image of the first map is a subset of the kernel of the second map, since $X-1$ maps to zero under the second map. If $P(X)$ is in the kernel of the second map, then 1 is a root of P and there exists $Q(X) \in \mathbb{Z}[X]$ such that $(X-1)Q(X) = P(X)$, i.e. the kernel of the second map is also a subset of the first map. So, for computing the right derived functors of $\mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}, -)$ for A, we have to compute the cohomology of the complex

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}[X], A) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}[X], A) \\ f &\longmapsto [P \mapsto f((X-1)P(X))] \end{aligned}$$

concentrated in the degrees 0 and 1. But since $\mathrm{Hom}_{\mathbb{Z}[X]}(\mathbb{Z}[X], A) \cong A$ as $\mathbb{Z}[X]$-module this complex translates into

$$\begin{aligned} A &\longrightarrow A \\ a &\longmapsto (X-1) \cdot a. \end{aligned}$$

This is exactly the second part of the claim. It remains to compute the cohomology groups. From the observations above we get

$$H^n(\mathbb{N}_0; A) = (R^n\mathrm{Hom}_{\mathbb{Z}[X]}(-, A))(\mathbb{Z}) = H^n(A \xrightarrow{X-1} A).$$

We then can immediately deduce that $H^n(\mathbb{N}_0; A) = 0$ for $n \in \mathbb{Z} \setminus \{0, 1\}$ and we get

$$\begin{aligned} H^0(\mathbb{N}_0; A) &= \ker(A \xrightarrow{X-1} A) \\ &= \{a \in A \mid X \cdot a = a\} \\ &= \{a \in A \mid n * a = a \text{ for all } n \in \mathbb{N}_0\} \\ &= A^{\mathbb{N}_0} \\ H^1(\mathbb{N}_0; A) &= \operatorname{coker}(A \xrightarrow{X-1} A) \\ &= A/\{a \in A \mid \text{ it exists } b \in A \text{ such that } (X-1) \cdot b = a\} \\ &= A_{\mathbb{N}_0}, \end{aligned}$$

what are exactly the claimed groups. Here, to avoid confusion, "$*$" denotes the operation from $\mathbb{N}_0$ on A. $\square$

Proposition 2.2.24.
Let G be a profinite group and $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. Then the double complex

$$K^{\bullet,\bullet} := C^\bullet(G,A) \xrightarrow{X-1} C^\bullet(G,A)$$

gives rise to two spectral sequences converging to the cohomology $\mathcal{H}^*_X(G,A)$:

$$\begin{aligned} \mathcal{H}^a_X(H^b(G,A)) &\Longrightarrow \mathcal{H}^{a+b}_X(G,A) \\ H^a(G,\mathcal{H}^b_X(A)) &\Longrightarrow \mathcal{H}^{a+b}_X(G,A). \end{aligned}$$

Proof.
Since for every $n \in \mathbb{Z}$ the double complex $K^{\bullet,\bullet}$ has at most two nonzero entries $K^{p,q}$ with $p+q=n$, this is shown in [Sta18, Tag 012X, Lemma 12.22.6]. $\square$

Lemma 2.2.25.
Let G be a profinite group and $f\colon A \to B$ be a morphism in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. Then the diagram

$$\begin{tikzcd} \mathcal{C}^n_X(G,A) \arrow[r, "C^n_X(G,f)"] \arrow[d, "\partial_A"] & C^n_X(G,B) \arrow[d, "\partial_B"] \\ \mathcal{C}^{n+1}_X(G,A) \arrow[r, "C^{n+1}_X(G,f)"] & C^{n+1}_X(G,B) \end{tikzcd}$$

is commutative for all $n \in \mathbb{N}_0$.

Proof.
Let $(x,y) \in C^n_X(G,A)$. Then compute

$$\begin{aligned} C^{n+1}_X(G,f)(\partial_A(x,y)) &= C^{n+1}_X(G,f)(\partial^n_A(x), (-1)^n(X-1)\cdot x + \partial^{n-1}_A(y)) \\ &= (f \circ (\partial^n_A(x)), f \circ ((-1)^n(X-1)\cdot x + \partial^{n-1}_A(y))), \\ \partial_B(C^n_X(G,f)(x,y)) &= \partial_B(f \circ x, f \circ y) \\ &= (\partial^n_B(f \circ x), (-1)^n(X-1)\cdot(f \circ x) + \partial^{n-1}_B(f \circ y)). \end{aligned}$$

Since the diagram

$$\begin{tikzcd} C^n(G,A) \arrow[r, "C^n(G,f)"] \arrow[d, "\partial_A"] & C^n(G,B) \arrow[d, "\partial_B"] \\ C^{n+1}(G,A) \arrow[r, "C^{n+1}(G,f)"] & C^{n+1}(G,B) \end{tikzcd}$$

is commutative for all $n \in \mathbb{N}_0$ (cf. [NSW15, Chapter I §3, p. 25]) we have $\partial^n_B \circ f = f \circ \partial^n_A$

for all $n \in \mathbb{N}_0$ and since, by assumption, f respects the action of $\mathbb{N}_0$ we have $f \circ (X-1) = (X-1) \circ f$. Using this in the above computation, we see that the diagram in fact commutes. $\square$

Lemma 2.2.26.
Let G be a profinite group and

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. Then, for every $n \in \mathbb{N}_0$, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^n_X(G,A) \arrow[r, "\mathcal{C}^n_X(G,\alpha)"] \arrow[d, "\partial_A"] & \mathcal{C}^n_X(G,B) \arrow[r, "\mathcal{C}^n_X(G,\beta)"] \arrow[d, "\partial_B"] & \mathcal{C}^n_X(G,C) \arrow[r] \arrow[d, "\partial_C"] & 0 \\
0 \arrow[r] & \mathcal{C}^{n+1}_X(G,A) \arrow[r, "\mathcal{C}^{n+1}_X(G,\alpha)"] & \mathcal{C}^{n+1}_X(G,B) \arrow[r, "\mathcal{C}^{n+1}_X(G,\beta)"] & \mathcal{C}^{n+1}_X(G,C) \arrow[r] & 0
\end{tikzcd}$$

is commutative with exact rows, i.e. the sequence

$$0 \longrightarrow \mathcal{C}^\bullet_X(G,A) \xrightarrow{\mathcal{C}^\bullet_X(G,\alpha)} \mathcal{C}^\bullet_X(G,B) \xrightarrow{\mathcal{C}^\bullet_X(G,\beta)} \mathcal{C}^\bullet_X(G,C) \longrightarrow 0$$

is exact.

Proof.
The commutativity is Lemma 2.2.25. Since A, B and C are discrete groups, we deduce from Corollary 2.1.12 that for all $n \in \mathbb{N}_0$ the sequence

$$0 \longrightarrow \mathcal{C}^n(G,A) \xrightarrow{\mathcal{C}^n(G,\alpha)} \mathcal{C}^n(G,B) \xrightarrow{\mathcal{C}^n(G,\beta)} \mathcal{C}^n(G,C) \longrightarrow 0$$

is exact. But since $\mathcal{C}^n_X(G,Z) = \mathcal{C}^n(G,Z) \oplus \mathcal{C}^{n-1}(G,Z)$ (where $\mathcal{C}^{-1}(G,Z) = 0$) and $\mathcal{C}^n_X(G,\eta) = \mathcal{C}^n(G,\eta) \oplus \mathcal{C}^{n+1}(G,\eta)$ for all $Z \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ and any morphism η in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$, we immediately deduce that the sequence

$$0 \longrightarrow \mathcal{C}^n_X(G,A) \xrightarrow{\mathcal{C}^n_X(G,\alpha)} \mathcal{C}^n_X(G,B) \xrightarrow{\mathcal{C}^n_X(G,\beta)} \mathcal{C}^n_X(G,C) \longrightarrow 0$$

is also exact. $\square$

Corollary 2.2.27.
Let G be a profinite group and

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. *Then, for every* $n \in \mathbb{N}_0$*, the diagram*

$$\begin{tikzcd} & \frac{\mathcal{C}_X^n(G,A)}{\mathrm{im}(\partial_A^{n-1})} \arrow[r, "\overline{\mathcal{C}_X^n(G,\alpha)}"] \arrow[d, "\overline{\partial_A^n}"] & \frac{\mathcal{C}_X^n(G,B)}{\mathrm{im}(\partial_B^{n-1})} \arrow[r, "\overline{\mathcal{C}_X^n(G,\beta)}"] \arrow[d, "\overline{\partial_B^n}"] & \frac{\mathcal{C}_X^n(G,C)}{\mathrm{im}(\partial_C^{n-1})} \arrow[r] \arrow[d, "\overline{\partial_C^n}"] & 0 \\ 0 \arrow[r] & \ker(\partial_A^{n+1}) \arrow[r, "\mathcal{C}_X^{n+1}(G,\alpha)"] & \ker(\partial_B^{n+1}) \arrow[r, "\mathcal{C}_X^{n+1}(G,\beta)"] & \ker(\partial_C^{n+1}) \end{tikzcd}$$

is commutative with exact rows. Here $\partial_Z^n \colon \mathcal{C}_X^n(G,Z) \to \mathcal{C}_X^{n+1}(G,Z)$ *denotes the* n*-th differential for* $Z \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$.

Proof.
The commutativity follows directly from Lemma 2.2.26. The upper row is the cokernel sequence of the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_X^{n-1}(G,A) \arrow[r, "\mathcal{C}_X^{n-1}(G,\alpha)"] \arrow[d, "\partial_A^{n-1}"] & \mathcal{C}_X^{n-1}(G,B) \arrow[r, "\mathcal{C}_X^{n-1}(G,\beta)"] \arrow[d, "\partial_B^{n-1}"] & \mathcal{C}_X^{n-1}(G,C) \arrow[r] \arrow[d, "\partial_C^{n-1}"] & 0 \\ 0 \arrow[r] & \mathcal{C}_X^{n}(G,A) \arrow[r, "\mathcal{C}_X^{n}(G,\alpha)"] & \mathcal{C}_X^{n}(G,B) \arrow[r, "\mathcal{C}_X^{n}(G,\beta)"] & \mathcal{C}_X^{n}(G,C) \arrow[r] & 0 \end{tikzcd}$$

and therefore it is exact. Similarly the lower row is the kernel sequence of the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_X^{n}(G,A) \arrow[r, "\mathcal{C}_X^{n}(G,\alpha)"] \arrow[d, "\partial_A"] & \mathcal{C}_X^{n}(G,B) \arrow[r, "\mathcal{C}_X^{n}(G,\beta)"] \arrow[d, "\partial_B"] & \mathcal{C}_X^{n}(G,C) \arrow[r] \arrow[d, "\partial_C"] & 0 \\ 0 \arrow[r] & \mathcal{C}_X^{n+1}(G,A) \arrow[r, "\mathcal{C}_X^{n+1}(G,\alpha)"] & \mathcal{C}_X^{n+1}(G,B) \arrow[r, "\mathcal{C}_X^{n+1}(G,\beta)"] & \mathcal{C}_X^{n+1}(G,C) \arrow[r] & 0, \end{tikzcd}$$

i.e. it is also exact. □

Lemma 2.2.28.
Let G *be a profinite group. The functors* $(\mathcal{H}_X^n(G,-))_n$ *then form a cohomological* δ*-functor, i.e. if*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is an exact sequence in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ *then, for every* $n \in \mathbb{N}_0$*, there is a group homomorphism*

$$\delta^n \colon \mathcal{H}_X^n(G,C) \longrightarrow \mathcal{H}_X^{n+1}(G,A)$$

such that the sequence

$$\cdots \longrightarrow \mathcal{H}^n_X(G,B) \longrightarrow \mathcal{H}^n_X(G,C) \xrightarrow{\delta^n} \mathcal{H}^{n+1}_X(G,A) \longrightarrow \mathcal{H}^{n+1}_X(G,B) \longrightarrow \cdots$$

is exact.

Proof.
The proof is the standard application for the snake lemma (cf. for example at [NSW15, (1.3.2) Theorem, Chapter I §3, p. 27]). We will give the proof here, to check that it really holds in this situation. For the snake lemma see [NSW15, (1.3.1) Snake Lemma, Chapter I §3, p. 25–26].
Let $n \in \mathbb{N}_0$. For $Z \in \mathcal{DIS}_{G,\mathbb{N}_0}$ let $\partial^n_Z \colon \mathcal{C}^n_X(G,Z) \to \mathcal{C}^{n+1}_X(G,Z)$ be the n-th differential. Corollary 2.2.27 says that the following commutative diagram has exact rows:

$$\begin{tikzcd}
& \frac{\mathcal{C}^n_X(G,A)}{\mathrm{im}(\partial^{n-1}_A)} \arrow[r, "\overline{\mathcal{C}^n_X(G,\alpha)}"] \arrow[d, "\overline{\partial^n_A}"] & \frac{\mathcal{C}^n_X(G,B)}{\mathrm{im}(\partial^{n-1}_B)} \arrow[r, "\overline{\mathcal{C}^n_X(G,\beta)}"] \arrow[d, "\overline{\partial^n_B}"] & \frac{\mathcal{C}^n_X(G,C)}{\mathrm{im}(\partial^{n-1}_C)} \arrow[r] \arrow[d, "\overline{\partial^n_C}"] & 0 \\
0 \arrow[r] & \ker(\partial^{n+1}_A) \arrow[r, "\mathcal{C}^{n+1}_X(G,\alpha)"] & \ker(\partial^{n+1}_B) \arrow[r, "\mathcal{C}^{n+1}_X(G,\beta)"] & \ker(\partial^{n+1}_C) &
\end{tikzcd}$$

Since the vertical kernels of the above diagram are the groups $\mathcal{H}^n_X(G,?)$ and the vertikal cokernels are the groups $\mathcal{H}^{n+1}_X(G,?)$ the snake lemma then says that there is an exact sequence[1]

$$\begin{tikzcd}
\mathcal{H}^n_X(G,A) \arrow[r] & \mathcal{H}^n_X(G,B) \arrow[r] & \mathcal{H}^n_X(G,C) \arrow[dll, "\delta^n"] \\
\mathcal{H}^{n+1}_X(G,A) \arrow[r] & \mathcal{H}^{n+1}_X(G,B) \arrow[r] & \mathcal{H}^{n+1}_X(G,C).
\end{tikzcd}$$

Doing this for all $n \in \mathbb{N}_0$ and connecting the sequences, this is exactly the claim. □

Lemma 2.2.29 (Adjunction of $\otimes$ and Hom)**.**
Let $R \to S$ be a homomorphism of commutative rings, X a R-module and Y, Z be S-modules. Then there holds

$$\mathrm{Hom}_R(Y \otimes_S Z, X) \cong \mathrm{Hom}_S(Y, \mathrm{Hom}_R(Z,X)),$$

where $\mathrm{Hom}_R(Z,X)$ *is a R-module via $(r \cdot f)(z) := r(f(z))$ for all $r \in R$, and $z \in Z$.*

Proof. [Sta18, Tag 05G3, Lemma 10.13.5] □

[1] The snake arrow is from `https://www.latex4technics.com/?note=93q`

Lemma 2.2.30.
Let G be a group. Then there holds

$$\mathbb{Z}[G][X] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X].$$

Proof.
First we want to note that the elements of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ can be written in the form $\sum_i (\eta_i \otimes X^i)$.
We will show the claim by showing that the homomorphism

$$\begin{aligned} \mathbb{Z}[G][X] &\longrightarrow \mathbb{Z}[G] \otimes \mathbb{Z}[X] \\ \sum_{i=0}^{n} \left(\sum_{g\in G} x_g^{(i)} \cdot g\right) X^i &\longmapsto \sum_{i=0}^{n} \left(\left(\sum_{g\in G} x_g^{(i)} \cdot g\right) \otimes X^i\right) \end{aligned}$$

is an isomorphism. But with the remark of the beginning, that every element of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ can be written in the form $\sum_i \eta_i \otimes X^i$, it is immediately clear that

$$\begin{aligned} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X] &\longrightarrow \mathbb{Z}[G][X] \\ \sum_{i=0}^{n} \left(\eta_i \otimes X^i\right) &\longmapsto \sum_{i=0}^{n} \eta_i X^i \end{aligned}$$

is the inverse map. □

Lemma 2.2.31.
Let G be an abelian profinite group. Then, for every $n \in \mathbb{N}$ the functor $\mathcal{H}_X^n(G, -)$ is effaceable, i.e. for every $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ there exists a $B \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ and a monomorphism $u\colon A \to B$ in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ such that $\mathcal{H}_X^n(G, u) = 0$.

Proof.
For this proof let I be an arbitrary product of $\mathbb{Q}/\mathbb{Z}$. Then I is an injective $\mathbb{Z}$-module. Note that every $\mathbb{Z}$-module can be embedded in such a module. Then also every object from $\mathcal{ABS}_{\mathrm{G},\mathbb{N}_0}$ can be embedded in a module of the form $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I)$. With Corollary 2.2.10 we than can conclude that every object of $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ can be embedded in a module of the form $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I)^{\delta}$. Therefore it is enough if we show

$$\mathcal{H}_X^n(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I)^{\delta}) = 0 \text{ for all } n > 0.$$

Set $T := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I)$. First we want to see that T^{δ} is an injective object in $\mathcal{DIS}_{\mathrm{G}}$. Recall from Lemma 2.2.11 that it is in fact an injective object in $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$.

We have

$$\begin{aligned} T &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X], I) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], I)). \end{aligned}$$

Here the second equation comes from Lemma 2.2.30 and the third from Lemma 2.2.29. Since $\mathbb{Z}[X] \cong \bigoplus_{\mathbb{N}_0} \mathbb{Z}$ we get $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], I) \cong \prod_{\mathbb{N}_0} I$ which then is again a product of $\mathbb{Q}/\mathbb{Z}$ (since I is so). Since then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], I)$ is an injective $\mathbb{Z}$-module, T is an injective object in $\mathcal{ABS}_{\mathrm{G}}$ and with Lemma 2.2.11 we then see that T^δ also is an injective object in $\mathcal{DIS}_{\mathrm{G}}$. In particular it is $H^b(G, T^\delta) = 0$ for all $b > 0$.
Next, we want to see that $(T^\delta)^G$ is an injective object in $\mathcal{ABS}_{\mathbb{N}_0}$. Recall from Corollary 2.2.8 that $(T^\delta)^G = T^G$. We then compute

$$\begin{aligned} (T^\delta)^G = T^G &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G][X], I)^G \\ &= \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G][X], I) \\ &= \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X], I) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], I)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], I). \end{aligned}$$

As above, the third equation comes from Lemma 2.2.30 and the fourth from Lemma 2.2.29. But this shows that $(T^\delta)^G$ is an injective object in $\mathcal{ABS}_{\mathbb{N}_0}$ and since $\mathcal{H}^a_X(-)$ coincides with $H^a(\mathbb{N}_0; -)$ (cf. Proposition 2.2.23), which itself is the a-th right derived functor of $(-)^{\mathbb{N}_0}$, we then get $\mathcal{H}^a_X((T^\delta)^G) = 0$ for $a > 0$ (cf. Remark 2.2.15).
Combining these two results, we obtain

$$\mathcal{H}^a_X(H^b(G, T^\delta)) = 0 \text{ if } a > 0 \text{ or } b > 0.$$

Since Proposition 2.2.24 says that

$$\mathcal{H}^a_X(H^b(G, T^\delta)) \Rightarrow \mathcal{H}^{a+b}_X(G, T^\delta)$$

we conclude that $\mathcal{H}^n_X(G, T^\delta) = 0$ if $n > 0$, as desired. □

Corollary 2.2.32.
Let G be a profinite group. Then the family of functors $(\mathcal{H}^n_X(-))_n$ from $\mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$ to **Ab** *forms a universal delta functor.*

Proof.

Lemma 2.2.28 says that $(\mathcal{H}^n_X(-))_n$ forms a delta functor and Lemma 2.2.31 says that the functors $\mathcal{H}^n_X(-)$ are effaceable for $n > 0$. This together shows that $(\mathcal{H}^n_X(-))_n$ is a universal delta functor. $\square$

Theorem 2.2.33.
Let G be a profinite group. Then we have

$$\mathcal{H}^n_X(G, A) = H^n(G, \mathbb{N}_0; A)$$

for all $n \in \mathbb{N}_0$ and $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$.

Proof.
Since $(H^n(G, \mathbb{N}_0; -))_n$ are the right derived functors of $(-)^{\mathcal{G},\mathbb{N}_0}$ this is a universal delta functor and since $(\mathcal{H}^n_X(G, -))_n$ is also an universal delta functor (cf. Corollary 2.2.32), it remains to check that they coincide in degree 0. For this, let $A \in \mathcal{DIS}_{\mathrm{G},\mathbb{N}_0}$. We have

$$H^0(G, \mathbb{N}_0; A) = A^{G,\mathbb{N}_0}$$

and

$$\begin{aligned} \mathcal{H}^0_X(G, A) &= H^0(\mathcal{C}^\bullet_X(G, A)) \\ &= \ker(A \xrightarrow{d^0} \mathcal{C}^1(G, A)) \cap \ker(A \xrightarrow{X-1} A) \\ &= A^G \cap A^{X=1} \end{aligned}$$

Since, by definition, $A^{X=1} = A^{\mathbb{N}_0}$ it follows immediately that $A^{G,\mathbb{N}_0} = A^G \cap A^{\mathbb{N}_0}$. $\square$

Next we want reformulate Proposition 2.2.17 with the above theorem, just to avoid confusions for latter applications.

Proposition 2.2.34.
Let G be a profinite group, $N \triangleleft G$ a closed, normal subgroup. Then there are two cohomological spectral sequences converging to $\mathcal{H}^n_X(G, -)$:

$$\begin{aligned} H^a(G/N, \mathcal{H}^b_X(N, A)) &\Longrightarrow \mathcal{H}^{a+b}_X(G, M; A) \\ \mathcal{H}^a_X(G/N, H^b(N, A)) &\Longrightarrow \mathcal{H}^{a+b}_X(G, M; A). \end{aligned}$$

Proof.
This is Proposition 2.2.17 using $H^n(G, \mathbb{N}_0; -) = \mathcal{H}^n_X(G, -)$ from Theorem 2.2.33. $\square$

As for the standard continuous cohomology (cf. [NSW15, (2.7.2) Lemma, Chapter II §7, p. 137]), we will also need a long exact sequence for $\mathcal{H}^*_X(G,-)$ in a slightly different setting as in Lemma 2.2.28.

Proposition 2.2.35.
Let G be a profinite group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence in $\mathcal{TOP}_{\mathrm{G},\mathbb{N}_0}$ such that the topology of A is induced by that of B and such that β has a continuous, set theoretical section. Then there are continuous homomorphisms

$$\delta^n \colon \mathcal{H}^n_X(G,C) \longrightarrow \mathcal{H}^{n+1}_X(G,A)$$

such that the sequence

$$\cdots \longrightarrow \mathcal{H}^n_X(G,B) \longrightarrow \mathcal{H}^n_X(G,C) \xrightarrow{\delta^n} \mathcal{H}^{n+1}_X(G,A) \longrightarrow \mathcal{H}^{n+1}_X(G,B) \longrightarrow \cdots$$

is exact.

Proof.
Algebraically this is exactly the same proof as Lemma 2.2.28. It then remains to check, that the occurring homomorphisms are continuous which is only for the δ^n a real question. But this can be answered using a topological version of the snake lemma, like [Sch99, Proposition 4, p. 133]. □

2.3 Some Homological Algebra

In this section we want to collect and prove some facts we will need later on.

Definition 2.3.1.
Let $C^\bullet$ be a complex of abelian groups and $n \in \mathbb{Z}$. Then we denote by $C^\bullet[n]$ the shift of this complex by n. This means, that for all $i \in \mathbb{Z}$ we have $C^i[n] = C^{i+n}$.

Lemma 2.3.2.
Let $Y^\bullet$ and $Z^\bullet$ be complexes of abelian groups and let $g^\bullet \colon Y^\bullet \to Z^\bullet$ be a morphism of complexes, such that every g^i is surjective. Then there is a canonical, surjective homomorphism

$$\ker(\mathrm{d}^i_Y) \cap \ker g^i \twoheadrightarrow H^i(\mathrm{Tot}(g^\bullet \colon Y^\bullet \to Z^\bullet)).$$

In particular, if all the g^i are bijective, we have

$$H^i(\mathrm{Tot}(g^\bullet\colon Y^\bullet \to Z^\bullet)) = 0.$$

Proof.
For the clarity of the presentation we write $H^i := H^i(\mathrm{Tot}(g^\bullet\colon Y^\bullet \to Z^\bullet))$. The i-th object of the total complex is $Y^i \times Z^{i-1}$ and the i-th differential $\mathrm{d}^i_{\mathrm{Tot}}$ is

$$\mathrm{d}^i_{\mathrm{Tot}} = \mathrm{d}^i_Y \circ \mathrm{pr}_1 \times (-1)^i g^i \circ \mathrm{pr}_1 + \mathrm{d}^{i-1}_Z \circ \mathrm{pr}_2.$$

We then compute

$$\ker \mathrm{d}^i_{\mathrm{Tot}} = \{(y,z) \in Y^i \times Z^{i-1} \mid \mathrm{d}^i_Y(y) = 0, (-1)^i g^i(y) + \mathrm{d}^{i-1}_Z(z) = 0\}$$

and

$$\begin{aligned} \operatorname{im} \mathrm{d}^{i-1}_{\mathrm{Tot}} = \{(y,z) \in Y^i \times Z^{i-1} \mid & \exists (y',z') \in Y^{i-1} \times Z^{i-2} : \\ & y = \mathrm{d}^{i-1}_Y(y'), z = (-1)^{i-1} g^{i-1}(y') + \mathrm{d}^{i-2}_Z(z')\} \end{aligned}$$

and we set $A^i := \ker \mathrm{d}^i_{\mathrm{Tot}}$ and $B^i := \operatorname{im} \mathrm{d}^{i-1}_{\mathrm{Tot}}$, i.e. we have $H^i = A^i/B^i$. There is a canonical homomorphism $\ker(\mathrm{d}^i_Y) \cap \ker g^i \to A^i$ sending y to $(y,0)$. Connecting with the canonical projection then gives a homomorphism $\ker(\mathrm{d}^i_Y) \cap \ker g^i \to H^i$.
So, let $(y,z) \in A^i$. Since g^{i-1} is surjective there is an $y' \in Y^{i-1}$ such that $(-1)^{i-1} g^{i-1}(y') = -z$, i.e. we have $(\mathrm{d}^{i-1}_Y(y'), -z) = \mathrm{d}^{i-1}_{\mathrm{Tot}}(y',0) \in B^i$ and therefore that the classes of (y,z) and $(y + \mathrm{d}^{i-1}_Y(y'), 0)$ in H^i coincide. So it remains to check $y + \mathrm{d}^{i-1}_Y(y') \in \ker(\mathrm{d}^i_Y) \cap g^i$. We already have $y, \mathrm{d}^{i-1}_Y(y') \in \ker(\mathrm{d}^i_Y)$ and

$$g^i(y) = (-1)^{i+1} \mathrm{d}^{i-1}_Z(z) = (-1)^i \mathrm{d}^{i-1}_Z((-1)^i g^{i-1}(y')) = -g^i \mathrm{d}^{i-1}_Y(y'),$$

i.e. $y + \mathrm{d}^{i-1}_Y(y') \in \ker g^i$. So, the class of $(y + \mathrm{d}^{i-1}_Y(y'), 0)$ in H^i is the image of $y + \mathrm{d}^{i-1}_Y(y')$ under the above map.
If now all the g^i are bijective, we have $\ker(g^i) = 0$ for every $i \in \mathbb{Z}$. Therefore it clearly is

$$H^i(\mathrm{Tot}(g^\bullet\colon Y^\bullet \to Z^\bullet)) = 0$$

for every $i \in \mathbb{Z}$. □

Lemma 2.3.3.
Let

$$0 \longrightarrow X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of abelian groups. Then the sequence

$$0 \longrightarrow X^\bullet \longrightarrow \operatorname{Tot}(Y^\bullet \to Z^\bullet) \longrightarrow \operatorname{Tot}(Y^\bullet/f^\bullet(X^\bullet) \to Z^\bullet) \longrightarrow 0$$

is also an exact sequence of complexes and for the cohomology we have

$$H^i(X^\bullet) \cong H^i(\operatorname{Tot}(g^\bullet \colon Y^\bullet \to Z^\bullet)).$$

Proof.
First note, that $f^i(X^i)$ is a subgroup of the kernel of

$$\mathrm{pr} \circ \mathrm{d}_Y \colon Y^i \longrightarrow Y^{i+1} \longrightarrow Y^{i+1}/f^{i+1}(X^{i+1})$$

and therefore we get a well defined homomorphism $\overline{\mathrm{d}_Y} \colon Y^i/f^i(X^i) \to Y^{i+1}/f^{i+1}(X^{i+1})$. We then have to check that for every $i \in \mathbb{Z}$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & X^i \arrow[r, "(f^i,0)"] \arrow[d, "\mathrm{d}_X"] & Y^i \times Z^{i-1} \arrow[r, "\mathrm{pr}\times\mathrm{id}_Z"] \arrow[d, "\mathrm{d}_Y\times g^i+\mathrm{d}_Z"] & Y^i/f^i(X^i) \times Z^{i-1} \arrow[r] \arrow[d, "\overline{\mathrm{d}_Y}\times\overline{g}^i+\mathrm{d}_Z"] & 0 \\ 0 \arrow[r] & X^{i+1} \arrow[r, "(f^{i+1},0)"] & Y^{i+1} \times Z^i \arrow[r, "\mathrm{pr}\times\mathrm{id}_Z"] & Y^{i+1}/f^{i+1}(X^{i+1}) \times Z^i \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows, where $\overline{g}^i \colon Y^i/f^i(X^i) \to Z^i$ is the from g^i and the given exact sequence induced homomorphism. We start with the exactness:
By assumption, for every $i \in \mathbb{Z}$ the homomorphism $f^i \colon X^i \to Y^i$ is injective and therefore the sequence

$$0 \longrightarrow X^i \xrightarrow{f^i} Y^i \xrightarrow{\mathrm{pr}} Y^i/f^i(X^i) \longrightarrow 0$$

is exact for every $i \in \mathbb{Z}$. But then also the sequence

$$0 \longrightarrow X^i \xrightarrow{(f^i,0)} Y^i \times Z^{i-1} \xrightarrow{\mathrm{pr}\times\mathrm{id}_Z} Y^i/f^i(X^i) \times Z^{i-1} \longrightarrow 0$$

is exact for every $i \in \mathbb{Z}$. To the commutativity:
By assumption we have $f^{i+1} \circ \mathrm{d}_X = \mathrm{d}_Y \circ f^i$ for all $i \in \mathbb{Z}$, which means that the first square of the above diagram commutes. For the second square, let $y \in Y^i$ and $z \in Z^{i-1}$. By definition, we have $\mathrm{pr}(\mathrm{d}_Y(y)) = \overline{\mathrm{d}_Y}(\mathrm{pr}(y))$ and $\overline{g}^i(\mathrm{pr}(y)) = g^i(y)$ and

therefore

$$(\mathrm{pr}(\mathrm{d}_Y(y)), g^i(y) + \mathrm{d}_Z(z)) = (\overline{\mathrm{d}_Y}(\mathrm{pr}(y)), \overline{g}^i(\mathrm{pr}(y)) + \mathrm{d}_Z(z)),$$

i.e. the second square commutes.
So the short sequence

$$0 \longrightarrow X^\bullet \longrightarrow \mathrm{Tot}(Y^\bullet \to Z^\bullet) \longrightarrow \mathrm{Tot}(Y^\bullet / f^\bullet(X^\bullet) \to Z^\bullet) \longrightarrow 0$$

is exact and we obtain from the long exact cohomology sequence that for every $i \in \mathbb{Z}$ the sequence

$$H^{i-1}(\mathrm{Tot}(\overline{g^\bullet}\colon Y^\bullet / f^\bullet(X^\bullet) \to Z^\bullet)) \longrightarrow H^i(X^\bullet) \longrightarrow H^i(\mathrm{Tot}(g^\bullet\colon Y^\bullet \to Z^\bullet))$$
$$\longrightarrow H^i(\mathrm{Tot}(\overline{g^\bullet}\colon Y^\bullet / f^\bullet(X^\bullet) \to Z^\bullet))$$

is exact. Lemma 2.3.2 says that the first and the last term in above sequence are 0 and therefore we get the claimed isomorphism. □

Corollary 2.3.4.
Let G be a profinite group, $A, B \in \mathcal{DIS}_{\mathrm{G}}$ and f a continuous endomorphism of B which respects the action of G such that the sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{f-1} B \longrightarrow 0$$

is exact. Then we have

$$H^i(G, A) = \mathcal{H}^i_f(G, B)$$

for all $i \geq 0$.

Proof.
This is just the above Lemma 2.3.3 with Corollary 2.1.12 and the notation from Definition 2.2.22. □

Corollary 2.3.5.
Let G be a profinite group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence in $\mathcal{TOP}_{\mathrm{G}}$, such that the topology of A is induced by that of B and sucht that β has a continuous, set theoretical section. Then the exact sequence of *complexes*

$$0 \longrightarrow C^\bullet_{\mathrm{cts}}(G,A) \xrightarrow{C^\bullet_{\mathrm{cts}}(G,\alpha)} C^\bullet_{\mathrm{cts}}(G,B) \xrightarrow{C^\bullet_{\mathrm{cts}}(G,\beta)} C^\bullet_{\mathrm{cts}}(G,C) \longrightarrow 0$$

(cf. Corollary 2.1.12*) induces*

$$A^G = H^0_{\mathrm{cts}}(G,A) \cong H^0(\mathrm{Tot}(C^\bullet_{\mathrm{cts}}(G,\beta)\colon C^\bullet_{\mathrm{cts}}(G,B) \to C^\bullet_{\mathrm{cts}}(G,C)))$$

and

$$C^G \to H^1_{\mathrm{cts}}(G,A) \cong H^1(\mathrm{Tot}(C^\bullet_{\mathrm{cts}}(G,\beta)\colon C^\bullet_{\mathrm{cts}}(G,B) \to C^\bullet_{\mathrm{cts}}(G,C))).$$

Proof.
This is an immediate consequence of the combination of the above Lemma 2.3.3 with Corollary 2.1.12. □

Now let's turn to some facts about projective limits.

Remark 2.3.6.
Note that $C^\bullet_{\mathrm{cts}}(G,-)$ commutes with projective limits, since the functors $\mathrm{Map}_{\mathrm{cts}}(G^n,-)$ and $(-)^G$ commute with projective limits, i.e. if $A = \varprojlim_n A_n$, then

$$C^\bullet_{\mathrm{cts}}(G,A) \cong \varprojlim_n C^\bullet_{\mathrm{cts}}(G,A_n).$$

Lemma 2.3.7.
Let G be a profinite group, $A \in \mathcal{TOP}_{\mathrm{G}}$ and let $(A_n)_n$ be an inverse system in $\mathcal{TOP}_{\mathrm{G}}$ such that $A = \varprojlim_n A_n$ in $\mathcal{TOP}_{\mathrm{G}}$. Let furthermore $f \in \mathrm{End}_{\mathrm{cts},G}(A)$, such that $f = \varprojlim f_n$ with $f_n \in \mathrm{End}_{\mathrm{cts},G}(A_n)$. Then there holds

$$\mathcal{C}^\bullet_f(G,A) \cong \varprojlim_n \mathcal{C}^\bullet_f(G,A_n).$$

Proof.
First we want to note, that for groups $X = \varprojlim X_n$ and $Y = \varprojlim Y_n$ always holds $X \times Y = \varprojlim_n (X_n \times Y_n)$.
This means that the objects of the two complexes $\mathcal{C}^\bullet_f(G,A)$ and $\varprojlim \mathcal{C}^\bullet_f(G,A_n)$ coincide, so it remains to check that the differentials do as well. If we denote the i-th object of $C^\bullet_{\mathrm{cts}}(G,A)$ by C^i and the differential by d^i then it suffices to check that the following

cube is commutative

$$\begin{tikzcd}
C^i \arrow[rr, "\mathcal{C}^i(G,f)"] \arrow[dr] \arrow[dd, "d^i"'] & & C^i \arrow[dr] \arrow[dd, "d^i" near start] & \\
& C^i \arrow[rr, "\varprojlim \mathcal{C}^i(G,f_n)" near start] \arrow[dd, "d^i" near start] & & C^i \arrow[dd, "d^i"] \\
C^{i+1} \arrow[rr, "\mathcal{C}^i(G,f)" near end] \arrow[dr] & & C^{i+1} \arrow[dr] & \\
& C^{i+1} \arrow[rr, "\varprojlim \mathcal{C}^i(G,f_n)"] & & C^{i+1}.
\end{tikzcd}$$

This is a direct consequence from the assumption $f = \varprojlim f_n$ and that $C^\bullet_{\text{cts}}(G, -)$ commutes with inverse limits. □

Lemma 2.3.8.
Let G be a profinite group and $(A_n)_n$ be an inverse system in $\mathcal{TOP}_\mathrm{G}$ such that the inverse system of complexes $(C^\bullet_{\text{cts}}(G, A_n))_n$ has surjective transition maps and let $A := \varprojlim_n A_n$. If $f \in \operatorname{End}_{\text{cts},G}(A)$ then also the system $(\mathcal{C}^\bullet_f(G, A_n))_n$ has surjective transition maps.

Proof.
By assumption, for every $k \in \mathbb{N}_0$, the transition map $C^k_{\text{cts}}(G, A_n) \to C^k_{\text{cts}}(G, A_{n-1})$ is surjective. But then also the transition map

$$\begin{tikzcd}
C^k_{\text{cts}}(G, A_n) \oplus C^{k-1}_{\text{cts}}(G, A_n) \arrow[r] \arrow[d, equal] & C^k_{\text{cts}}(G, A_{n-1}) \oplus C^{k-1}_{\text{cts}}(G, A_{n-1}) \arrow[d, equal] \\
\mathcal{C}^k_f(G, A_n) & \mathcal{C}^k_f(G, A_{n-1})
\end{tikzcd}$$

is surjective, since it's the direct sum of two surjective maps. □

Definition 2.3.9.
An inverse system (of abelian groups) $(X_n)_{n\in\mathbb{N}}$ is called **Mittag-Leffler** (**ML**) if for any $n \in \mathbb{N}$, there is an $m \geq n$ such that the image of the transition maps $X_k \to X_n$ coincide for all $k \geq m$ (cf. [NSW15, p. 138]).
An inverse system (of abelian groups) $(X_n)_{n\in\mathbb{N}}$ is called **Mittag-Leffler zero** (**ML-zero**) if for any $n \in \mathbb{N}$ there is an $m \geq n$ such that the transition map $X_k \to X_n$ is zero for all $k \geq m$ (cf. [NSW15, p. 139]).
A morphism $(X_n)_n \to (Y_n)_n$ of inverse systems is called **Mittag-Leffler isomorphism** (**ML-isomorphism**) if the corresponding systems of kernels and cokernels

are ML-zero.

By $\varprojlim^r$ we denote the r-th right derived functor of $\varprojlim$.

Proposition 2.3.10.

Let (X_n) und (Y_n) be inverse systems of abelian groups.

1. *If (X_n) has surjective transition maps, then it is ML.*

2. *If (X_n) is ML then $\varprojlim_n^r X_n = 0$ for all $r \geq 0$.*

3. *If $f_n\colon X_n \to Y_n$ is a ML-isomorphism then for all $i \geq 0$ the homomorphism*

$$\varprojlim_n^i f_n\colon \varprojlim_n^i X_n \longrightarrow \varprojlim_n^i Y_n$$

is an isomorphism.

Proof.

1. Let $\alpha_{nm}\colon X_m \to X_n$ denote the transition map for $m \geq n$. Then it is $\mathrm{im}(\alpha_{nm}) = X_n$ for all $m \geq n$, i.e. the system X_n is ML.

2. [NSW15, Chapter II §7, (2.7.4) Proposition, p. 140]

3. First note that the systems $(\ker(f_n))$ and $(\mathrm{coker}(f_n))$ are ML since they are ML-zero and therefore it holds $\varprojlim_n^i \ker(f_n) = 0$ and $\varprojlim_n^i \mathrm{coker}(f_n) = 0$ for all $i \geq 0$. Now consider the following two short exact sequences

$$0 \longrightarrow \ker(f_n) \longrightarrow X_n \longrightarrow \mathrm{im}(f_n) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{im}(f_n) \longrightarrow Y_n \longrightarrow \mathrm{coker}(f_n) \longrightarrow 0.$$

Taking inverse limits together with the assumption that both systems $(\ker(f_n))$ and $(\mathrm{coker}(f_n))$ are ML-zero then gives the long exact sequences

$$\dashrightarrow 0 \longrightarrow \varprojlim_n^i X_n \longrightarrow \varprojlim_n^i \mathrm{im}(f_n) \longrightarrow 0 \longrightarrow \varprojlim_n^{i+1} X_n \longrightarrow \varprojlim_n^{i+1} \mathrm{im}(f_n) \dashrightarrow$$

$$\dashrightarrow 0 \longrightarrow \varprojlim_n^i \mathrm{im}(f_n) \longrightarrow \varprojlim_n^i Y_n \longrightarrow 0 \longrightarrow \varprojlim_n^{i+1} \mathrm{im}(f_n) \longrightarrow \varprojlim_n^{i+1} Y_n \dashrightarrow .$$

Since $\mathrm{im}(f_n) \to Y_n$ is the canonical inclusion, the second sequence implies $\varprojlim_n^i \mathrm{im}(f_n) = \varprojlim_n^i Y_n$ for all $i \geq 0$. Together with the first sequence, this then says that

$$\varprojlim_n^i f_n\colon \varprojlim_n^i X_n \xrightarrow{\cong} \varprojlim_n^i \mathrm{im}(f_n) = \varprojlim_n^i Y_n$$

is an isomorphism for all $i \geq 0$.

$\square$

Proposition 2.3.11.
Let $(X_n^\bullet)$ and $(Y_n^\bullet)$ be inverse systems of complexes of abelian groups such that the transition maps $X_{n+1}^i \to X_n^i$ and $Y_{n+1}^i \to Y_n^i$ are surjective for all $i \in \mathbb{Z}$ and $n \geq 0$.

1. *For all $i \in \mathbb{Z}$ we get a short exact sequence*
$$0 \longrightarrow \varprojlim_n^1 H^{i-1}(X_n^\bullet) \longrightarrow H^i(\varprojlim_n X_n^\bullet) \longrightarrow \varprojlim_n H^i(X_n^\bullet) \longrightarrow 0.$$

2. *Let $(f_n^\bullet)\colon (X_n^\bullet) \to (Y_n^\bullet)$ be a morphism of inverse systems of complexes. If the induced map on cohomology $H^i(f_n^\bullet)\colon H^i(X_n^\bullet) \to H^i(Y_n^\bullet)$ is a ML-isomorphism for all $i \in \mathbb{Z}$, then $\varprojlim_n(f_n^\bullet)\colon \varprojlim_n X_n^\bullet \to \varprojlim_n Y_n^\bullet$ is a quasi isomorphism.*

Proof.

1. [Soc80, Chapter 3, Proposition 1, p. 531; Corollary1.1, p. 535–536]

2. From the first part of the proposition we obtain for every $i \in \mathbb{Z}$ a commutative diagram with exact rows
$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_n^1 H^{i-1}(X_n^\bullet) \arrow[r] \arrow[d, "\varprojlim_n^1 H^{i-1}(f_n^\bullet)"] & H^i(\varprojlim_n X_n^\bullet) \arrow[r] \arrow[d, "H^i(\varprojlim_n f_n^\bullet)"] & \varprojlim_n H^i(X_n^\bullet) \arrow[r] \arrow[d, "\varprojlim_n H^i(f_n^\bullet)"] & 0 \\
0 \arrow[r] & \varprojlim_n^1 H^{i-1}(Y_n^\bullet) \arrow[r] & H^i(\varprojlim_n Y_n^\bullet) \arrow[r] & \varprojlim_n H^i(Y_n^\bullet) \arrow[r] & 0.
\end{tikzcd}$$
The assumption that $H^i(f_n^\bullet)$ is a ML-isomorphism for all $i \in \mathbb{Z}$ then says that the left and the right horizontal maps in the above diagram are isomorphisms (cf. Proposition 2.3.10). The 5-Lemma then implies that also $H^i(\varprojlim_n f_n^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$, i.e. $\varprojlim_n f_n^\bullet$ is a quasi isomorphism.

$\square$

Remark 2.3.12.
Since isomorphisms of inverse systems are always ML-isomorphisms, the above Proposition also states, that if $(f_n^\bullet)\colon (X_n^\bullet) \to (Y_n^\bullet)$ is a quasi isomorphism of inverse systems of complexes, for which the transition maps $X_{n+1}^i \to X_n^i$ and $Y_{n+1}^i \to Y_n^i$ are surjective for all $i \in \mathbb{Z}$ and $n \geq 0$, then also $\varprojlim_n(f_n^\bullet)\colon \varprojlim_n X_n^\bullet \to \varprojlim_n Y_n^\bullet$ is a quasi isomorphism.

Remark 2.3.13.

In the above Proposition 2.3.11 and Remark 2.3.12 one cannot easily drop the assumption that the transition maps are surjective. In the following we will give an example of two inverse systems of complexes which are quasi isomorphic, but their projective limits are not. In our opinion, because of this example, in the proof of [Sch06, Theorem 2.2.1, p. 702–705] right before [Sch06, Proposition 2.2.7, p. 703–705], their should be an explanation why it really is enough to prove this proposition.
The first inverse system of complexes we consider is the complex wich is everywhere 0. This complex and its projective limit complex clearly have cohomology equal to 0. The nontrivial inverse system of complexes is the system consisting of

$$\cdots \longrightarrow 0 \longrightarrow p^n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

for every n. The transition maps of the inverse system $(p^n\mathbb{Z})_n$, $(\mathbb{Z})_n$ and $(\mathbb{Z}/p^n\mathbb{Z})_n$ are the inclusion for the first two and the canonical projection for the last one. Then one immediately obtains that the inverse system $(p^n\mathbb{Z})_n$ has not surjective transition maps. Since this complex is exact, we have for every $n \in \mathbb{N}$ a quasi isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & p^n\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Taking projective limits of these inverse systems of complexes then gives

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

But $\mathbb{Z} \neq \mathbb{Z}_p$ and therefore the upper complex has a nontrivial cohomology groups equal to $\mathbb{Z}_p/\mathbb{Z}$ while the lower complex still has cohomology equal to zero.

We will end this section with a proposition with strongly reminds on the universal coefficient theorem in the sense of [Che09, Theorem 3.21, p. 13] , but for which there is no proper reference. Instead of explaining how consisting statements transfer to our, we decided us to give a straightforward proof of the proposition.

Proposition 2.3.14.

Let R be a commutative ring with unit, $\mathcal{C}^\bullet$ be a cochain complex of R-modules and V

a flat R-module. Then there holds

$$H^*(\mathcal{C}^\bullet) \otimes_R V \cong H^*(\mathcal{C}^\bullet \otimes_R V).$$

Proof.
This is [Nek07, (3.4.4) Proposition, p. 66–67] $\square$

Chapter 3

Lubin-Tate (φ, Γ)-modules

The goal in this chapter is to generalize the equivalence of categories from [Sch17] in a way similar to the original result [FO10, Theorem 4.22, p. 82] for (ϕ, Γ)-modules in the cyclotomic case. Namely, if $K|L|\mathbb{Q}_p$ are finite extensions, we want to establish an equivalence of categories between the category of continuous $\mathcal{O}_L$-representations of the absolute Galois group G_K and a yet to be defined category of étale (φ_L, Γ_K)-modules. In order to do this, we will go through the book [Sch17], starting around section 1.7 and explain how one transfers the results to the relative case of a finite extension of L. Unfortunately we have to permute the order of [Sch17], since the construction of the coefficient ring involves some facts, which in loc. cit. are important only later on. One more useful source will be [Sch11].

3.1 Preparations and Notations

Let p be a prime number and let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of the p-adic numbers $\mathbb{Q}_p$ and let as usual $\mathbb{Z}_p$ be the integral p-adic numbers. Each finite extension of $\mathbb{Q}_p$ is considered to be a subfield of $\overline{\mathbb{Q}_p}$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ with respect to the valuation v_p with $v_p(p) = 1$ and let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$.
Let furthermore $L|\mathbb{Q}_p$ be a finite extension, d_L its degree over $\mathbb{Q}_p$, $\mathcal{O}_L$ the ring of integers, $\pi_L \in \mathcal{O}_L$ a prime element, k_L the residue class field, $q_L = p^r$ its cardinality, L^{ur} the maximal unramified extension of $\mathbb{Q}_p$ in L with ring of integers $\mathcal{O}_{L^{\mathrm{ur}}}$.
Let furthermore $K|L$ be a finite extension, d_K its degree over $\mathbb{Q}_p$, $\mathcal{O}_K$ its ring of integers, $\pi_K \in \mathcal{O}_K$ a prime element, k_K the residue class field, q_K its cardinality, K^{ur} the maximal unramified extension of $\mathbb{Q}_p$ in K with ring of integers $\mathcal{O}_{K^{\mathrm{ur}}}$.
We will denote the absolute Galois groups of $\mathbb{Q}_p$, L and K by $G_{\mathbb{Q}_p}$, G_L and G_K respectively.

By $W(\cdot)_L$ we will denote **ramified Witt vectors** (cf. [Sch17, Section 1.1, p. 6–21]). Roughly speaking, these are standard Witt vectors tensored with $\mathcal{O}_L$ (cf. [Sch17, Proposition 1.1.26, p. 23–24]).
A **perfectoid** field $\mathcal{K} \subseteq \mathbb{C}_p$ is a complete field, such that its value group $|\mathcal{K}^\times|$ is dense in $\mathbb{R}_+^\times$ and which satisfies $(\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^p = \mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K}$ (cf. [Sch17, p. 42]).
Let $\mathcal{K}$ be a perfectoid field. The **tilt** $\mathcal{K}^\flat$ of $\mathcal{K}$ is the fraction field of the ring

$$\mathcal{O}_{\mathcal{K}^\flat} := \varprojlim_{x \mapsto x^{q_L}} \mathcal{O}_\mathcal{K}/\varpi\mathcal{O}_\mathcal{K},$$

where ϖ is an element in $\mathcal{O}_\mathcal{K}$ such that $|\varpi| \geq |\pi_L|$. In fact, this definition is independent from the choice of the element ϖ (cf. [Sch17, Lemma 1.4.5, p. 43–44]). The field $\mathcal{K}^\flat$ is perfect and complete and has characteristic p (cf. [Sch17, Proposition 1.4.7, p. 45]). Moreover, the field $\mathbb{C}_p^\flat$ is algebraically closed (cf. [Sch17, Proposition 1.4.10, p. 46–47]). The theory of perfectoid fields was originally established by Peter Scholze (cf. [Sch11]) but Schneider's book covers all of the theory we do need here.
Let from now on, as in [Sch17, Definition 1.3.2, p. 29], $\phi \in R[\![X_1, \dots X_n]\!]$ be a fixed Frobenius power series associated to π_L, i.e. we have

$$\begin{aligned}\phi(X) &\equiv \pi_L X \bmod \deg 2\\ \phi(X) &= X^{q_L} \bmod \pi_L\mathcal{O}_L[\![X]\!].\end{aligned}$$

Let furthermore $\mathcal{G}_\phi \in \mathcal{O}_L[\![X, Y]\!]$ be the Lubin-Tate formal group which belongs to ϕ (cf. [Sch17, Proposition 1.3.4, p. 31]). For $a \in \mathcal{O}_L$ denote by $[a]_\phi \in \mathcal{O}_L[\![X]\!]$ the corresponding endomorphism of $\mathcal{G}_\phi$ (cf. [Sch17, Proposition 1.3.6, p. 32]). Note that we then have $[a]_\phi(X) \equiv aX \bmod \deg 2$ and $[\pi_L]_\phi = \phi$ (loc. cit.). We then set $\mathfrak{M} := \{x \in \overline{\mathbb{Q}_p} \mid |x| < 1\}$ and obtain that the operation

$$\begin{aligned}\mathcal{O}_L \times \mathfrak{M} &\longrightarrow \mathfrak{M}\\ (a, x) &\longmapsto [a]_\phi(z)\end{aligned}$$

makes $\mathfrak{M}$ into an $\mathcal{O}_L$-module (cf. [Sch17, p. 33]). Then, for every $a \in \mathcal{O}_L$, we can view $[a]_\phi$ as endomorphism of $\mathfrak{M}$ and therefore are able to define

$$\mathcal{G}_{\phi,n} := \ker([\pi_L^n]_\phi \colon \mathfrak{M} \to \mathfrak{M}) = \{x \in \mathfrak{M} \mid [\pi_L^n]_\phi(x) = 0\}.$$

Note that $(\mathcal{G}_{\phi,n})_n$ is via $[\pi_L]_\phi$ an inverse system and we let

$$\mathcal{T}\mathcal{G}_\phi := \varprojlim_n \mathcal{G}_{\phi,n}$$

be the projective limit of this system. (cf. [Sch17, p. 50]). $\mathcal{T}\mathcal{G}_\phi$ is also called the **Tate module** of the group $\mathcal{G}_\phi$. From [Sch17, Proposition 1.3.10, p. 34] we can deduce that $\mathcal{T}\mathcal{G}_\phi$ is a free $\mathcal{O}_L$-module of rank one.
Following [Sch17, (1.3.9), p. 33] we let $L_n = L(\mathcal{G}_\phi[\pi_L^n])$ and $L_\infty = \cup_n L_n$. Denote as there the Galois group $\mathrm{Gal}(L_\infty|L)$ by Γ_L, set $\Gamma_{L_n|L} = \mathrm{Gal}(L_n|L)$ and $H_L = \mathrm{Gal}(\overline{\mathbb{Q}_p}|L_\infty)$. Define furthermore $K_n := K(\mathcal{G}_\phi[\pi_L^n]) = KL_n$ and $K_\infty := \cup_n K_n = KL_\infty$ as well as $\Gamma_K = \mathrm{Gal}(K_\infty|K)$ and $H_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}|K_\infty)$. These definitions can be summarized in the following diagram:

$$\begin{tikzcd}
 & \overline{\mathbb{Q}_p} \arrow[dl, dash, "H_L"'] \arrow[dr, dash, "H_K"] \arrow[ddddl, dash, bend right=60, "G_L"'] \arrow[dddr, dash, bend left=50, "G_K"] & \\
 & & K_\infty \arrow[dd, dash, "\Gamma_K"'] \\
L_\infty \arrow[urr, dash] \arrow[dd, dash, "\Gamma_L"] & & \\
 & & K \\
L \arrow[urr, dash] & &
\end{tikzcd}$$

Remark 3.1.1.
The group Γ_L is isomorphic to $\mathcal{O}_L^\times$ via the Lubin-Tate character χ_{LT}.
Furthermore, Γ_L acts continuously on $\mathcal{T}\mathcal{G}_\phi$ via χ_L, i.e. for all $\gamma \in \Gamma_L$ and $t \in \mathcal{T}\mathcal{G}_\phi$ we have

$$\gamma \cdot t = \chi_L(\gamma) \cdot t = [\chi_L(\gamma)]_\phi(t).$$

Proof.
For the first assertion see [Sch17, (1.3.12),p. 36], the second follows immediately from [Sch17, (1.3.11),p. 34–35] and is also stated at [Sch17, (1.4.17),p. 51]. □

Remark 3.1.2.
One can view Γ_K as an open subgroup of Γ_L.
If, in addition, $K|L$ is unramified, then we have $\Gamma_K \cong \Gamma_L$.

Proof.
G_K is a subgroup of G_L, it is closed (since it corresponds to a subfield of $\overline{\mathbb{Q}_p}|L$) and it's index is $(G_L : G_K) = [K : L]$, which is finite, i.e. G_K is an open subgroup of G_L.

Furthermore because of $K_\infty = KL_\infty$ it is $H_K = H_L \cap G_K$, which is the kernel of the canonical homomorphism $G_K \hookrightarrow G_L \twoheadrightarrow G_L/H_L$. Since the canonical projection is, by definition, open, this homomorphism is continuous and open and therefore induces a continuous and open inclusion

$$\Gamma_K \cong G_K/H_K \hookrightarrow G_L/H_L \cong \Gamma_L.$$

Let now $K|L$ be unramified. Since, for all $n \in \mathbb{N}$ the finite extension $L_n|L$ is Galois and totally ramified (cf. [Sch17, Proposition 1.3.12, p. 35–36]), the extension $K_n = KL_n|K$ also is Galois and totally ramified. The Galois group of $K_n|K$ then is isomorphic to $\mathrm{Gal}(L_n|L_n \cap K)$ but since $L_n|L$ is totally ramified and $K|L$ is, by assumption, unramified it clearly is $L_n \cap K = L$ and therefore we obtain

$$\mathrm{Gal}(L_n|L) \cong \mathrm{Gal}(K_n|K).$$

From this we deduce the claimed isomorphism

$$\Gamma_L = \varprojlim_n \mathrm{Gal}(L_n|L) \cong \varprojlim_n \mathrm{Gal}(K_n|K) = \Gamma_K.$$

□

3.2 THE COEFFICIENT RING

We first want to recall the definition of the coefficient ring used in [Sch17] and then deduce the coefficient ring in our general case.
Before going into the construction of the coefficient ring, we want to recall the ring

$$\mathscr{A}_L := \varprojlim_n \mathcal{O}_L/\pi_L^n \mathcal{O}_L((X)),$$

from [Sch17, p. 75]. This ring will be prototypical for our coefficients if we can bring the variable X to life. Schneider then explains, that $\mathscr{A}_L$ carries an action from Γ_L by

$$\begin{aligned}\Gamma_L \times \mathscr{A}_L &\longrightarrow \mathscr{A}_L\\ (\gamma, f) &\longmapsto f([\chi_L(\gamma)]_\phi(X)).\end{aligned}$$

and an injective $\mathcal{O}_L$-algebra endomorphism

$$\begin{aligned}\varphi_L \colon \mathscr{A}_L &\longrightarrow \mathscr{A}_L\\ f &\longmapsto f([\pi_L]_\phi(X))\end{aligned}$$

(cf. [Sch17, p. 78]). At [Sch17, p. 79] Schneider defines a **weak topology** on $\mathscr{A}_L$, for which the $\mathcal{O}_L[\![X]\!]$-submodules

$$U_m := X^m \mathcal{O}_L[\![X]\!] + \pi_L^m \mathscr{A}_L$$

form a fundamental system of open neighbourhoods of $0 \in \mathscr{A}_L$. He makes several observations for $\mathscr{A}_L$ which we want to summarize in the following proposition.

Proposition 3.2.1.

1. *As $\varphi_L(\mathscr{A}_L)$-module $\mathscr{A}_L$ is free with basis $1, X, \ldots, X^{q_L - 1}$.*

2. *With respect to the weak topology $\mathscr{A}_L$ is a complete Hausdorff topological $\mathcal{O}_L$-algebra.*

3. *The endomorphism φ_L and the Γ_L-action are continuous for the weak topology.*

Proof.

1. [Sch17, Proposition 1.7.3, p. 78].

2. [Sch17, Lemma 1.7.6, p. 79–80].

3. [Sch17, Proposition 1.7.8, p. 80–82].

□

Let us now head towards the definition of our coefficient ring. An important part is, that one can find an element $\omega \in \mathcal{O}_{\mathbb{C}_p^\flat}$, such that $X \mapsto \omega$ defines an inclusion $k_L((X)) \hookrightarrow \mathbb{C}_p^\flat$. As in [Sch17, p. 50] we denote the image of this inclusion by $\mathbf{E}_L$ and we want to recall from loc. cit. that $\mathbf{E}_L$ is a complete nonarchimedean discretely valued field, with uniformizer ω and residue class field k_L. Let in addition $\mathbf{E}_L^+$ denote the ring of integers inside $\mathbf{E}_L$. Furthermore, $\mathbf{E}_L$ carries a continuous operation by Γ_L, for wich we have $\gamma \cdot \omega = [\chi_L(\gamma)]_\phi(\omega) \bmod \pi_L$ (cf. [Sch17, Lemma 1.4.15, p. 51]). By raising elements to its q_L-th power, it is clear that $\mathbf{E}_L$ also carries a Frobenius homomorphism, which is continuous and the reduction modulo p of φ_L. Let furthermore $\mathbf{E}_L^{\mathrm{sep}}$ denote the separable closure of $\mathbf{E}_L$ inside $\mathbb{C}_p^\flat$ and let $\mathbf{E}_L^{\mathrm{sep},+}$ denote the integral closure of $\mathbf{E}_L^+$ inside $\mathbf{E}_L^{\mathrm{sep}}$. A really helpful fact is the following:

Theorem 3.2.2.
The Galois group $\mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}} | \mathbf{E}_L)$ *is isomorphic to H_L.*

Proof.
This is [Sch17, Section 1.6, p. 68–75] and [Sch17, Theorem 1.6.7, p. 73–74] in particular.

□

Then Schneider lifts ω to $W(\mathbf{E}_L)_L \subseteq W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$ and calls this lift ω_ϕ (cf. [Sch17, Section 2.1, p. 84–98; in particular p. 93]). Here one cannot just take the Teichmüller lift, because one wants that the lift fulfills the following relations

$$\begin{aligned}\mathrm{Fr}(\omega_\phi) &= [\pi_L]_\phi(\omega_\phi)\\ \gamma \cdot \omega_\phi &= [\chi_L(\gamma)]_\phi(\omega_\phi)\end{aligned}$$

for all $\gamma \in \Gamma_L$ and where Fr is the Frobenius on $W(\mathbb{C}_p^\flat)_L$ (cf. [Sch17, Lemma 2.1.13, p. 92–93] for the Frobenius and [Sch17, Lemma 2.1.15, p. 95] for the Γ_L-action). Similar to the construction of $\mathbf{E}_L$, sending X to ω_ϕ then defines an inclusion $\mathscr{A}_L \hookrightarrow W(\mathbf{E}_L)_L$ (cf. [Sch17, p. 94]). In Particular, it gives us a commutative square (loc. cit.)

$$\begin{tikzcd} \mathscr{A}_L \arrow[r, "X\mapsto\omega_\phi"] \arrow[d] & W(\mathbf{E}_L)_L \arrow[d] \\ k_L((X)) \arrow[r, "X\mapsto\omega"] & \mathbf{E}_L. \end{tikzcd}$$

Following Schneider, we let $\mathbf{A}_L$ denote the image of the inclusion $\mathscr{A}_L \hookrightarrow W(\mathbf{E}_L)_L$. In addition, define

$$\mathbf{A}_L^+ \coloneqq \mathcal{O}_L[\![\omega_\phi]\!] = \mathbf{A}_L \cap W(\mathbf{E}_L^+)_L.$$

He then also endows $\mathbf{A}_L$ with a weak topology, induced by that from $W(\mathbb{C}_p^\flat)_L$, and observes that the isomorphism $\mathscr{A}_L \cong \mathbf{A}_L$ then is topological for the weak topologies on both sides (cf. [Sch17, Proposition 2.1.16, p. 95–96]). Furthermore, he proves that this topological isomorphism respects the Γ_L-actions on both sides, where $\mathbf{A}_L$-carries a Γ_L-action induced from the G_L-action of $W(\mathbb{C}_p^\flat)_L$ (cf. [Sch17, p. 94]) and states that what is φ_L on $\mathscr{A}_L$ is the Frobenius on $\mathbf{A}_L$, which again is induced from the Frobenius on $W(\mathbb{C}_p^\flat)_L$ (cf. [Sch17, Proposition 2.1.16, p. 95–96]). We therefore denote the Frobenius on $\mathbf{A}_L$ also by φ_L. An immediate consequence then is, that the Γ_L-action and φ_L are continuous on $\mathbf{A}_L$.

This then is the coefficient ring in for Schneiders (φ_L, Γ_L)-modules (cf. [Sch17, Definition 2.2.6, p. 100–101]) but since we want to establish (φ, Γ)-modules over a finite extension $K|L$ as it was done in the classical way (cf. [FO10, Definition 4.21, p. 81]) for finite extensions of $\mathbb{Q}_p$, we transfer this construction to our situation. Let for this $\mathbf{A}_L^{\mathrm{nr}} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ be the maximal unramified extension of $\mathbf{A}_L$ inside $W(\mathbf{E}_L^{\mathrm{sep}})_L$. In particular [Sch17, Lemma 3.1.3, p. 112–113] says that for every finite, separable extension $F|\mathbf{E}_L$ inside $\mathbf{E}_L^{\mathrm{sep}}$, there exists a unique ring $\mathbf{A}_L(F) \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ containing $\mathbf{A}_L$ such that $\mathbf{A}_L^{\mathrm{nr}}$ is the colimit of the family $\mathbf{A}_L(F)$. Additionally Schneider defines the ring $\mathbf{A}$ as the closure of $\mathbf{A}_L^{\mathrm{nr}}$ inside $W(\mathbf{E}_L^{\mathrm{sep}})$ with respect to the π_L-adic topology and observes (cf. [Sch17, p. 113 and Remark 3.14, p. 114])

$$\mathbf{A} \cong \varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}.$$

He then also states that both, $\mathbf{A}_L^{\mathrm{nr}}$ and $\mathbf{A}$, have an action from G_L, that the Frobenius on $W(\mathbf{E}_L^{\mathrm{sep}})$ preserves both rings, that they are discrete valuation rings with prime element π_L, where $\mathbf{A}$ is even complete and that their residue class field is $\mathbf{E}_L^{\mathrm{sep}}$ (cf. [Sch17, p. 113–114]). In fact, the G_L-action on both $\mathbf{A}_L^{\mathrm{nr}}$ and $\mathbf{A}$ is continuous for the weak topologies, since the G_L action on $W(\mathbb{C}_p^\flat)_L$ is continuous for the weak topology (cf. [Sch17, Lemma 1.4.13, p. 48–49] and [Sch17, Lemma 1.5.3, p. 65–66]) and both, the weak topology and the G_L action on $\mathbf{A}_L^{\mathrm{nr}}$ respectively $\mathbf{A}$, are induced form $W(\mathbb{C}_p^\flat)_L$. In addition, then every subgroup of G_L acts continuously on $\mathbf{A}_L^{\mathrm{nr}}$ and $\mathbf{A}$. Furthermore we have the relation (cf. [Sch17, Lemma 3.1.6, p. 115–116])

$$(\mathbf{A})^{H_L} = \mathbf{A}_L.$$

This then leads us to the definition

$$\mathbf{A}_K \coloneqq (\mathbf{A})^{H_K}.$$

In addition, define

$$\begin{aligned} \mathbf{A}^+ &\coloneqq \mathbf{A} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L \\ \mathbf{A}_L^{\mathrm{nr},+} &\coloneqq \mathbf{A}_L^{\mathrm{nr}} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L \\ \mathbf{A}_K^+ &\coloneqq \mathbf{A}_{K|L} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L. \end{aligned}$$

Then, since by definition it is $\mathbf{A}_L \subseteq \mathbf{A}_{K|L} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$, the ring $\mathbf{A}_{K|L}$ is a complete nonarchimedean discrete valuation ring with prime element π_L and the restriction of the Frobenius from $W(\mathbf{E}_L^{\mathrm{sep}})_L$ gives a ring endomorphism of $\mathbf{A}_{K|L}$ which then also commutes with φ_L (cf. [Sch17, Lemma 3.1.3, p. 112–113]). We will denote this endomorphism by $\varphi_{K|L}$. Furthermore, since $\mathbf{A}$ carries an action from G_L and therefore also one from G_K, the ring $\mathbf{A}_{K|L}$ carries an action from Γ_K. Next, we want to define a weak topology an $\mathbf{A}_{K|L}$, deduce some properties and see that $\varphi_{K|L}$ and the action from Γ_K are continuous for this topology.

Definition 3.2.3.
The **weak topology** on any of the rings $\mathbf{A}$, $\mathbf{A}_L^{\mathrm{nr}}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L$ is defined as the induced topology of the weak topology of $(W(\mathbb{C}_p^\flat))_L$ (for the latter see [Sch17, p. 64–65]).

Remark 3.2.4.
The weak topology on $W(\mathbb{C}_p^\flat)_L$ is complete and Hausdorff (cf. [Sch17, Lemma 1.5.5, p. 67–68]*) and $W(\mathbb{C}_p^\flat)_L$ is a topological ring with respect to its weak topology (cf.* [Sch17, Lemma 1.5.4, p. 66–67]*). Therefore, the induced topology on any of the rings $\mathbf{A}$, $\mathbf{A}_L^{\mathrm{nr}}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L$ is Hausdorff and these rings are topological rings.*

The question now is, wether $\varphi_{K|L}$ and the action from Γ_K are continuous for the weak topology on $\mathbf{A}_{K|L}$. For this, we want to recall a well-known fact.

Lemma 3.2.5.
Let X and Y be topological spaces, $f\colon X \to Y$ be a continuous map and let $Z \subseteq Y$ be a subspace with $\mathrm{im}(f) \subseteq Z$. Then $f\colon X \to Z$ is continuous.

Proposition 3.2.6.
The from $W(\mathbf{E}_L^{\mathrm{sep}})_L$ induced Γ_K-action and the induced Frobenius $\varphi_{K|L}$ on $\mathbf{A}_{K|L}$ are continuous.

Proof.
This now is an immediate consequence of Lemma 3.2.5 and the fact, that G_L acts continuously on $W(\mathbf{E}_L^{\mathrm{sep}})_L$ (cf. [Sch17, Lemma 1.5.3, p. 65–66]) as well as that Fr is continuous on $W(\mathbf{E}_L^{\mathrm{sep}})$ with respect to the weak topology:
Since the maps

$$G_L \times \mathbf{A}_{K|L} \longrightarrow G_L \times W(\mathbf{E}_L^{\mathrm{sep}})_L \longrightarrow W(\mathbf{E}_L^{\mathrm{sep}})_L$$

and

$$\mathbf{A}_{K|L} \hookrightarrow W(\mathbf{E}_L^{\mathrm{sep}})_L \xrightarrow{\mathrm{Fr}} W(\mathbf{E}_L^{\mathrm{sep}})_L$$

are continuous as composite maps of continuous maps and their image is inside $\mathbf{A}_{K|L}$ (for the latter see [Sch17, Lemma 3.1.3, p. 112–113]) the claim follows. $\square$

We want to end this section by fixing some notation, defining weak topologies on modules over any of the above rings and calculating the residue class field of $\mathbf{A}_{K|L}$. We start by fixing notation and denote by the quotient field of $\mathbf{A}_L$. Similarly we denote by $\mathbf{B}$, $\mathbf{B}_{K|L}$ and $\mathbf{B}_L^{\mathrm{nr}}$ the quotient fields of $\mathbf{A}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L^{\mathrm{nr}}$, respectively. Furthermore, set $\mathbf{E}_{K|L} := (\mathbf{E}_L^{\mathrm{sep}})^{H_K}$ and let $\mathbf{E}_{K|L}^+$ denote the integral closure of $\mathbf{E}_L^+$ inside $\mathbf{E}_L$. In Lemma 3.2.13 we will see that $\mathbf{E}_{K|L}$ is the residue class field of $\mathbf{A}_{K|L}$. Beforehand, we define weak topologies for modules.

Lemma 3.2.7.
Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. If $k, l \in \mathbb{N}$ such that $R^k \twoheadrightarrow M$ and $R^l \twoheadrightarrow M$ are surjective homomorphisms, then the induced quotient topologies on M coincide (where R^k and R^l carry the product topology of the weak topology on R).

Proof. This is [Kle16, Lemma 3.2.2 (i), p. 100–102]. There, in fact, is no proof for $\mathbf{A}_{K|L}$, but in his proof, the author only uses that the coefficient ring is a topological ring with respect to the weak topology, what we stated in the above Remark 3.2.4. □

Definition 3.2.8.
Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. The **weak topology** on M is defined as the quotient topology for any surjective homomorphism $R^k \twoheadrightarrow M$, where R^k carries the product topology of the weak topology on R.

Lemma 3.2.9.
Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. Then M with its weak topology is a topological R-module and if $M = M_1 \oplus M_2$, then the weak topology on M coincides with the direct product of the weak topologies on the M_1 and M_2.
Furthermore, if N is another finitely generated R-module and $f\colon M \to N$ is an R-module homomorphism, then f is continuous with respect to the weak topologies on both M and N.

Proof. This is [Kle16, Lemma 3.2.2 (ii)-(iv), p. 100–102]. Again, there is no proof for $\mathbf{A}_{K|L}$, but the property used is that of a discrete valuation ring, which $\mathbf{A}_{K|L}$ also fulfills. □

Proposition 3.2.10 (Relative Ax-Sen-Tate)**.**
Let $\mathcal{K}$ be a nonarchimedean valued field of characteristic 0, $\overline{\mathcal{K}}$ an algebraic closure with completion $\mathcal{C}$ and $\mathcal{L}|\mathcal{K}$ a Galois extension within $\overline{\mathcal{K}}$ with completion $\widehat{\mathcal{L}}$. Let furthermore $H \leq \mathrm{Gal}(\mathcal{L}|\mathcal{K})$ be a closed subgroup. Then it holds

$$(\widehat{\mathcal{L}})^H = (\mathcal{L}^H)^\wedge.$$

Proof.
This is an immediate consequence of the usual Ax-Sen-Tate theorem (cf. [FO10, Proposition 3.8, p. 43–44]):
Since $\mathcal{L}|\mathcal{K}$ is algebraic, $\overline{\mathcal{K}}$ is also an algebraic closure for $\mathcal{L}$ and then we deduce (loc.

cit.)

$$\mathcal{C}^{G_{\mathcal{L}}} = \widehat{\mathcal{L}}.$$

Infinite Galois theory then says that we have $H = \mathrm{Gal}(\mathcal{L}|\mathcal{L}^H) \cong G_{\mathcal{L}^H}/G_{\mathcal{L}}$. Together with Ax-Sen-Tate we then deduce

$$(\mathcal{L}^H)^\wedge = \mathcal{C}^{G_{\mathcal{L}^H}} = (\mathcal{C}^{G_{\mathcal{L}}})^H = (\widehat{\mathcal{L}})^H.$$

□

For our purposes the following integral version of the above Relative Ax-Sen-Tate Theorem will be the interesting one.

Corollary 3.2.11.
Let $\mathcal{K}$ be a nonarchimedean valued field of characteristic 0, $\overline{\mathcal{K}}$ an algebraic closure with completion $\mathcal{C}$ and $\mathcal{L}|\mathcal{K}$ a Galois extension within $\overline{\mathcal{K}}$ with completion $\widehat{\mathcal{L}}$. Denote by $\mathcal{O}_?$ the ring of integers of any of the above fields. Let furthermore $H \leq \mathrm{Gal}(\mathcal{L}|\mathcal{K})$ be a closed subgroup. Then it holds

$$(\mathcal{O}_{\widehat{\mathcal{L}}})^H = ((\mathcal{O}_{\mathcal{L}})^H)^\wedge.$$

Proof.
For an element $x \in \mathcal{C}$ we have

$$x \in (\mathcal{O}_{\widehat{\mathcal{L}}})^H \Leftrightarrow x \in (\widehat{\mathcal{L}})^H \text{ with } |x| \leq 1 \overset{3.2.10}{\Leftrightarrow} x \in (\mathcal{L}^H)^\wedge \text{ with } |x| \leq 1 \Leftrightarrow x \in ((\mathcal{O}_{\mathcal{L}})^H)^\wedge,$$

where the last equivalence comes from the fact that the integers of the completion are the completion of the integers. □

Lemma 3.2.12.
It holds $(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = \mathbf{A}_{K|L}$.

Proof.
This is a direct consequence of the above Corollary 3.2.11. This namely says that

$$\mathbf{A}_{K|L} = (\mathbf{A})^{H_K} = ((\mathbf{A}_L^{\mathrm{nr}})^{H_K})^\wedge.$$

But since $(\mathbf{A}_L^{\mathrm{nr}})^{H_K}|\mathbf{A}_L$ is finite and $\mathbf{A}_L$ is complete, $(\mathbf{A}_L^{\mathrm{nr}})^{H_K}$ itself is complete, i.e. it is

$$(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = ((\mathbf{A}_L^{\mathrm{nr}})^{H_K})^\wedge = \mathbf{A}_{K|L}.$$

□

Lemma 3.2.13.
$\mathbf{E}_{K|L}$ is the residue class field of $\mathbf{A}_{K|L}$.

Proof.
We have an exact sequene

$$0 \longrightarrow \mathbf{A}_L^{\mathrm{nr}} \xrightarrow{\cdot \pi_L} \mathbf{A}_L^{\mathrm{nr}} \longrightarrow \mathbf{A}_L^{\mathrm{nr}}/\pi_L \mathbf{A}_L^{\mathrm{nr}} \longrightarrow 0.$$

By taking H_K-invariants and using $(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = \mathbf{A}_{K|L}$ from Lemma 3.2.12 we obtain the exact sequence

$$0 \longrightarrow \mathbf{A}_{K|L} \xrightarrow{\cdot \pi_L} \mathbf{A}_{K|L} \longrightarrow (\mathbf{E}_L^{\mathrm{sep}})^{H_K} \longrightarrow H^1(H_K, \mathbf{A}_L^{\mathrm{nr}}).$$

Since $\mathbf{B}_L^{\mathrm{nr}}|\mathbf{B}_L$ is unramified, and therefore also tamely ramified, we get from [NSW15, (6.1.10) Theorem, p. 342–342] that $\mathbf{A}_L^{\mathrm{nr}}$ is a cohomologically trivial H_L-module. Therefore the right term in the latter sequence is equal to zero and we get the exact sequence

$$0 \longrightarrow \mathbf{A}_{K|L} \xrightarrow{\cdot \pi_L} \mathbf{A}_{K|L} \longrightarrow \mathbf{E}_{K|L} \longrightarrow 0$$

which ends the proof. □

3.3 Concrete description of Weak Topologies

As the title says, the goal of this chapter is to give a concrete description of both, the ring $\mathbf{A}_{K|L}$ and its weak topology. We will start with the topology and first we want the recall the description of the weak topology of $\mathbf{A}_L$ and recall that a similar description holds true on $W(\mathbb{C}_p^\flat)_L$

Remark 3.3.1.
[Sch17, Proposition 2.1.16 (i), p. 95–96] *says that the weak topology $\mathbf{A}_L$ has an analogous description as the description above. Concretely, a fundamental system of open neighbourhoods of 0 for the weak topology on $\mathbf{A}_L$ is given by*

$$\omega_\phi^m \mathbf{A}_L^+ + \pi_L^m \mathbf{A}_L, \ m \geq 1.$$

Remark 3.3.2.
A fundamental system of open neighbourhoods of 0 for the weak topology on $W(\mathbb{C}_p^\flat)_L$ is given by the $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$-submodules

$$\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L, \ m \geq 1.$$

Proof.
Because of $|\Phi_0(\omega_\phi)|_\flat = |\omega|_\flat = |\pi_L|^{q_L/q_L-1} < 1$ (cf. [Sch17, Lemma 2.1.13 (i), p. 92–93] for the first equality and [Sch17, Lemma 1.4.14, p. 50] for the second) this is exactly [Sch17, Remark 2.1.5 (ii), p. 86–87]. □

The above remarks raise hope, that a similar description holds true on intermediate rings. In fact, in the following Proposition we will show, that the above description of the weak topology on $\mathbf{A}_L$ extends to unramified, integral extensions. Its proof is a generalization of [Sch17, Proposition 2.1.16 (i), p. 95–96].

Proposition 3.3.3.
Let $B|\mathbf{B}_L$ be an unramified extension, $A \subseteq B$ the integral closure of $\mathbf{A}_L$ in B and set $A^+ := A \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L$.
Then the family

$$\omega_\phi^m A^+ + \pi_L^m A, \ m \geq 1$$

of A^+-submodules of A forms a fundamental system of open neighbourhoods of 0 for the weak topology on A.

Proof.
Since we have $\omega_\phi^m A^+ \subseteq \omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$ and $\pi_L^m A \subseteq \pi_L^m W(\mathbb{C}_p^\flat)_L$ for all $m \geq 1$, we also get

$$\omega_\phi^m A^+ + \pi_L^m A \subseteq (\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L) \cap A$$

for all $m \geq 1$, i.e. the topology on A generated by the family $(\omega_\phi^m A^+ + \pi_L^m A)_m$ is finer then the topology induced from $W(\mathbb{C}_p^\flat)_L$.
To see that it is also coarser, let $E|\mathbf{E}_L$ be the residue class field of A and E^+ be the integral closure of $\mathbf{E}_L^+$ in E and consider the following families of $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$-submodules of $W(\mathbb{C}_p^\flat)_L$:

$$V_{n,m} := \left\{(b_0, b_1, \dots) \in W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \mid b_0, \dots, b_{m-1} \in \omega^n \mathcal{O}_{\mathbb{C}_p^\flat}\right\},$$
$$U_{n,m} := \left\{(b_0, b_1, \dots) \in W(\mathbb{C}_p^\flat)_L \mid b_0, \dots, b_{m-1} \in \omega^n \mathcal{O}_{\mathbb{C}_p^\flat}\right\}.$$

These are introduced in [Sch17, Section 1.5, p. 64–68] to define the weak topology on $W(\mathbb{C}_p^\flat)_L$. In particular, the $U_{n,m}$ give a fundamental system of open neighbourhoods of 0 in $W(\mathbb{C}_p^\flat)_L$ (loc. cit.) and the $V_{n,m}$ give one of $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$. Since ω_ϕ is topologically nilpotent (cf. [Sch17, Lemma 2.1.6, p. 87]) we can find for any $k \in \mathbb{N}$ an element $n \in \mathbb{N}$ such that $\omega_\phi^n \in V_{k,m}$. But since $\Phi_0(\omega_\phi) = \omega$, i.e. $\omega_\phi = (\omega, \dots)$, the condition $\omega_\phi^n \in V_{k,m}$ implies $n \geq k$. Therefore we can find an increasing sequence of natural numbers $m \leq l_1 < \cdots < l_m$ such that

$$\omega_\phi^{q_L^{l_i}} \in V_{q_L^{l_{i-1}+1},m} \quad \text{for all } 2 \leq i \leq m.$$

Since A^+ only contains positive powers of ω_ϕ, this then implies, that for all $2 \leq i \leq m$ we have

$$\omega_\phi^{q_L^{l_i}} A^+ \subseteq V_{q_L^{l_{i-1}+1},m}.$$

We will now show that

$$U_{q_L^{l_m},m} \cap A \subseteq \omega_\phi^m A^+ + \pi_L^m A.$$

For this let $f_m \in U_{q_L^{l_m},m} \cap A$. We then have

$$\Phi_0(f_m) \in \omega^{q_L^{l_m}} \mathcal{O}_{\mathbb{C}_p^\flat} \cap E = \omega^{q_L^{l_m}} E^+.$$

Since by [Sch17, Lemma 3.1.3 (b), p. 112–113] the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\mathrm{pr}"'] & & W(E)_L \arrow[dl, "\Phi_0"] \\ & E & \end{tikzcd}$$

commutes, we can find $g_m \in \omega_\phi^{q_L^{l_m}} A^+$ and $f_{m-1} \in A$ such that

$$f_m = g_m + \pi_L f_{m-1}.$$

Recall $\omega_\phi^{q_L^{l_m}} A^+ \subseteq V_{q_L^{l_{m-1}+1},m}$ from above and obtain

$$\pi_L f_{m-1} = f_m - g_m \in (U_{q_L^{l_m},m} + V_{q_L^{l_{m-1}+1},m}) \cap A = U_{q_L^{l_{m-1}+1},m} \cap A.$$

Then [Sch17, Proposition 1.1.18 (i), p. 16–17] says that, if $f_{m-1} = (b_0, b_1, \dots)$ for some $b_j \in \mathbb{C}_p^\flat$ then we have $\pi_L f_{m-1} = (0, b_0^{q_L}, b_1^{q_L}, \dots)$. This then immediately implies $f_{m-1} \in U_{q_L^{l_{m-1}},m} \cap A$. This means that we can do a decreasing induction for $m \geq i \geq 1$ and find for every such i elements $g_i \in \omega_\phi^{q_L^{l_i}} A^+$ and $f_{i-1} \in A$ such that

$$f_i = g_i + \pi_L f_{i-1}.$$

Putting all this together, we get

$$f_m = \sum_{i=1}^{m} \pi_L^{m-i} g_m + \pi_L^m f_0.$$

In particular we have

$$\sum_{i=1}^{m} \pi_L^{m-i} g_m \in \omega_\phi^{q_L^{l_1}} A^+ \subseteq \omega_\phi^m A^+.$$

Therefore we have $f_m \in \omega_\phi^m A^+ + \pi_L^m A$ which was exactly the statement we wanted to see to end the proof. □

Corollary 3.3.4.
A fundamental system of open neighbourhoods of 0 *for the weak topology on* $\mathbf{A}_{K|L}$ *(resp.* $\mathbf{A}_L^{\mathrm{nr}}$*) is given by the* $\mathbf{A}_{K|L}^+$*- (resp.* $\mathbf{A}_L^{\mathrm{nr},+}$*-) submodules*

$$\begin{aligned} &\omega_\phi^m \mathbf{A}_{K|L}^+ + \pi_L^m \mathbf{A}_{K|L},\ m \geq 1,\ \textit{respectively} \\ &\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m \mathbf{A}_L^{\mathrm{nr}},\ m \geq 1. \end{aligned}$$

Proof.
This is an application of Proposition 3.3.3. □

Proposition 3.3.5.
The weak topology on $\mathbf{A}_{K|L}$ *coincides with the weak topology of* $\mathbf{A}_{K|L}$ *considered as* $\mathbf{A}_L$*-module.*

Proof.
If $(u_i)_i$ is an $\mathbf{A}_L$-basis of $\mathbf{A}_{K|L}$, then $(\omega_\phi^k u_i)_i$ is so for all $k \geq 0$. Therefore $\mathbf{A}_{K|L}$ has an $\mathbf{A}_L$-basis consisting of elements of $\mathbf{A}_{K|L}^+$. The claim then follows from the above Corollary 3.3.4 together with Corollary 3.3.1. □

Proposition 3.3.6.
The canonical inclusion $\mathbf{A}_{K|L} \hookrightarrow \mathbf{A}$ *is a topological embedding. Furthermore, for every* $n \in \mathbb{N}$ *the induced inclusion* $\mathbf{A}_{K|L}/\pi_L^n \mathbf{A}_{K|L} \hookrightarrow \mathbf{A}/\pi_L^n \mathbf{A}$ *is a topological embedding as well.*

Proof.
Because of

$$\mathbf{A}_{K|L} \cap \mathbf{A} = \mathbf{A}_{K|L} \cap \mathbf{A} \cap W(\mathcal{O}_{\mathbb{C}_p^\flat})_L = \mathbf{A}_{K|L} \cap W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$$

the first part of the assertion follows from the definition of the weak topology. The second then follows from the commutative diagram

$$\begin{tikzcd} \mathbf{A}_{K|L} \arrow[r, hook] \arrow[d, two heads] & \mathbf{A} \arrow[d, two heads] \\ \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L} \arrow[r, hook] & \mathbf{A}/\pi_L^n\mathbf{A}. \end{tikzcd}$$

□

Proposition 3.3.7.
The weak topology on $\mathbf{A}$ coincides with the topology of the projective limit $\varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}}$ where each factor carries the quotient topology of the weak topology on $\mathbf{A}_L^{\mathrm{nr}}$.
Moreover, a fundamental system of open neighbourhoods of 0 for the weak topology on $\mathbf{A}$ is given by the sets

$$\omega_\phi^m\mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m\mathbf{A},\ m \geq 1.$$

Note that, by definition, $\mathbf{A}^+ = \mathbf{A}_L^{\mathrm{nr},+}$.

Proof.
For this proof, we will refer to the latter topology of the Proposition's formulation as the *projective limit topology.*
As in the above Proposition 3.3.6 the inclusion $\mathbf{A}_L^{\mathrm{nr}} \hookrightarrow \mathbf{A}$ clearly is a topological embedding and since the diagram

$$\begin{tikzcd} \mathbf{A}_L^{\mathrm{nr}} \arrow[r, hook] \arrow[d, two heads] & \mathbf{A} \arrow[d, two heads] \\ \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}} \arrow[r, equal] & \mathbf{A}/\pi_L^n\mathbf{A}. \end{tikzcd}$$

for every $n \in \mathbb{N}$ is commutative, the quotient topology on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}}$ with respect to the weak topology on $\mathbf{A}_L^{\mathrm{nr}}$ coincides with its quotient topology with respect to the weak topology on $\mathbf{A}$. Therefore the canonical projections

$$\mathbf{A} = \varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}} \twoheadrightarrow \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}}$$

are continuous for the weak topology on $\mathbf{A}$. This means that the weak topology of $\mathbf{A}$ is finer than its projective limit topology.
From Proposition 3.3.3 we deduce that a fundamental system of open neighbourhoods of 0 for the quotient topology of the weak topology on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n\mathbf{A}_L^{\mathrm{nr}}$ is given by the

sets

$$\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^n \mathbf{A}_L^{\mathrm{nr}},\ m \geq 1.$$

Then the sets

$$\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^n \mathbf{A},\ m, n \geq 1$$

form a fundamental system of open neighbourhoods of 0 for the projective limit topology on $\mathbf{A}$. But clearly the sets with $m = n$ define the same topology. Since the weak topology is defined by the sets

$$\left(\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L\right) \cap \mathbf{A},\ m \geq 1$$

(cf. Remark 3.3.2) and we clearly have

$$\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m \mathbf{A} \subseteq \left(\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L\right) \cap \mathbf{A}$$

for all $m \geq 1$, the projective limit topology is finer than the weak topology. □

Lemma 3.3.8.
Let k be a finite field and $E|k((X))$ be a finite, separable extension. Then there exists a finite extension $\kappa|k$ such that $E \cong \kappa((Y))$.

Proof.
This is [Kup15, Lemma 1.38, p. 20]. □

Lemma 3.3.9.
Let $k'|k$ be an extension of finite fields and $k'((Y))|k((X))$ be a finite, separable extension. Then the Y-adic and the X-adic topologies on $k'((Y))$ coincide.
In particular, there exists a $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds

$$X^{ln} k'[\![Y]\!] \subseteq Y^{ln} k'[\![Y]\!] \subseteq X^n k'[\![Y]\!].$$

Proof.
Since $k[\![X]\!]$ is a discrete valuation ring with respect to its X-adic topology and $k'[\![Y]\!]$ is so as well with respect to its Y-adic topology, we deduce from usual ramification theory, that there exists a $l \in \mathbb{N}$ such that

$$Y^l k'[\![Y]\!] = X k'[\![Y]\!].$$

Since $Y k'[\![Y]\!]$ is the maximal ideal of $k'[\![Y]\!]$ it clearly is $X k'[\![Y]\!] \subseteq Y k'[\![Y]\!]$ and therefore we get for all $n \in \mathbb{N}$

$$X^{ln}k'[\![Y]\!] \subseteq Y^{ln}k'[\![Y]\!] \subseteq X^n k'[\![Y]\!].$$

$\square$

Lemma 3.3.10.
Let $E|\mathbf{E}_L$ be a finite and separable extension. Then the subspace topology on E induced from the topology of $\mathbb{C}_p^\flat$ coincides with the extension from the ω-adic topology on $\mathbf{E}_L$. Note that the latter topology is the ω-adic topology on E, due to the above Lemma 3.3.9.
In particular, the integral closure E^+ of $\mathbf{E}_L^+$ inside E consists of exactly those elements of E whose absolute value in $\mathbb{C}_p^\flat$ is less or equal to 1.

Proof.
We denote the absolute value induced from $\mathbb{C}_p^\flat$ by $|\cdot|_\flat$ as in [Sch17, Lemma 1.4.6, p. 44–45] and we use the identifications $E \cong \kappa((Y))$ as well as $\mathbf{E}_L \cong k_L((X))$ (cf. Lemma 3.3.8), where $\kappa|k_L$ is a finite extension.
The maximal unramified intermediate field of $\kappa((Y))|k_L((X))$ is $\kappa((X))$ and therefore it exists a $l \in \mathbb{N}$ and $g_i \in \kappa[\![X]\!]$ for $0 \leq i < l$ with $X \mid g_i$ and $X^2 \nmid g_0$ such that (cf. [Ser79, Chapter I, §6, Proposition 17, p. 19])

$$\sum_{i=0}^{l-1} g_i Y^i + Y^l = 0.$$

Since $|X|_\flat < 1$ and $|x|_\flat = 1$ for $x \in \kappa$ (in particular, every nonzero element coming from a finite field has absolute value 1 in $\mathcal{O}_{\mathbb{C}_p^\flat}$ with respect to $|\cdot|_\flat$) we have $|g_i|_\flat \leq 1$ for all $0 \leq i < l$ and we can deduce

$$|Y^l|_\flat \leq \max_{0 \leq i < l} |g_i|_\flat |Y^i|_\flat \leq \max_{0 \leq i < l} |Y^i|_\flat$$

and therefore $|Y|_\flat \leq 1$. Furthermore, since we have $Y^l\kappa[\![Y]\!] = X\kappa[\![Y]\!]$ we can find a $g \in \kappa[\![Y]\!]$ such that $Y^l = Xg$ and since $|Y|_\flat \leq 1$ we then deduce $|g|_\flat \leq 1$ and

$$|Y^l|_\flat = |X|_\flat |g|_\flat \leq |X|_\flat < 1.$$

But this then immediately implies

$$|Y|_\flat < 1.$$

Since $X \mid g_0$ and $X^2 \nmid g_0$ it is $|g_0|_\flat = |X|_\flat$ and because of $X \mid g_i$ for all $0 \leq i < l$ we also have

$$|g_0|_\flat \geq |g_i|_\flat \quad \text{for all} \quad 0 < i < l.$$

Since $|Y|_\flat < 1$ we deduce from the above

$$|g_0|_\flat > |g_i|_\flat |Y^i|_\flat \quad \text{for all} \quad 0 < i < l$$

and therefore

$$|Y^l|_\flat = |g_0|_\flat = |X|_\flat$$

because $|\cdot|_\flat$ is a nonarchimedean absolute value.

Denote by $|\cdot|$ the extension of the absolute value of $\mathbf{E}_L$ (which corresponds with the ω-adic topology) to E. Then we deduce from [Ser79, Chapter 2, §2, Corollary 4, p. 29] that

$$|Y^l| = |\mathrm{Nor}(Y)|_\flat,$$

where Nor denotes the norm of the extension $\kappa((Y))|\kappa((X))$. From the polynomial we started with we then can deduce $\mathrm{Nor}(Y) = g_0$ and therefore

$$|Y^l| = |g_0|_\flat = |X|_\flat.$$

This means that $|\cdot|$ and $|\cdot|_\flat$ coincide on E.

From the identification above we deduce $E^+ = \kappa[\![Y]\!]$. But since $|Y|_\flat < 1$, these are exactly the elements of E whose absolute value is less or equal to 1. □

Corollary 3.3.11.

$\mathbf{E}_{K|L}$ is, with respect to the topology induced from $\mathbb{C}_p^\flat$, a complete, nonarchimedean discretely valued field of characteristic p with residue class field k_K and ring of integers $\mathbf{E}^+_{K|L}$.

Lemma 3.3.12.

Let X be a topological space and $(Y_n)_n$ a family of subsets of X with $Y_n \subseteq Y_{n+1}$. Set $Y \coloneqq \varinjlim_n Y_n = \bigcup_n Y_n$. Then, the subset topology on Y coincides with the final topology of the inductive limit with respect to the subset topologies on the Y_n.

Proof.

First we show that the canonical injections $f_n \colon Y_n \hookrightarrow Y$ are continuous for the subset topology on Y. This then implies that the subspace topology on Y is coarser than the projective limit topology since the latter is the finest such that all injections f_n are continuous (cf. [Bou89a, Chapter I, §2.4, Proposition 6, p. 32]).

Let $U \subseteq Y$ be open and $V \subseteq X$ open such that $U = V \cap Y$. Then it is

$$f_n^{-1}(U) = U \cap Y_n = V \cap V \cap Y_n = V \cap Y_n,$$

i.e. $f_n^{-1}(U) \subseteq Y_n$ is open.
It is left to show, that the subspace topology is finer then the direct limit topology. For this, let $U \subseteq Y$ be open with respect to the direct limit topology, i.e. it is $U = \bigcup_n f_n^{-1}(U)$, where for every $n \in \mathbb{N}$ it exists an open $V_n \subseteq X$ such that $f^{-1}(U) = V_n \cap Y_n$. We set $V := \bigcup V_n$ and claim $U = V \cap Y$. To see this, let $u \in U$. Then it exists $n \in \mathbb{N}$ such that $u \in V_n \cap Y_n$ and in particular $u \in V$. Conversely let $u \in V \cap Y$. Then, by definition, there exist $n_1, n_2 \in \mathbb{N}$ such that $u \in V_{n_1}$ and $u \in Y_{n_2}$. For $n := \max\{n_1, n_2\}$ we then deduce $u \in V_n \cap Y_n$ and therefore $u \in U$. □

Proposition 3.3.13.
The integral closure $\mathbf{E}_L^{\mathrm{sep},+}$ of $\mathbf{E}_L^+$ inside $\mathbf{E}_L^{\mathrm{sep}}$ consists of exactly those elements with absolute value $|\cdot|_\flat$ less or equal to 1.
Furthermore, the topology on $\mathbf{E}_L^{\mathrm{sep}}$ induced from $\mathbb{C}_p^\flat$ coincides with the final topology with respect to the colimit

$$\mathbf{E}_L^{\mathrm{sep}} = \bigcup_{\substack{E|\mathbf{E}_L \\ \text{fin, sep}}} E$$

where each E carries the topology induced from $\mathbb{C}_p^\flat$.
In particular, the $\mathbf{E}_L^{\mathrm{sep},+}$-submodules

$$\omega^n \mathbf{E}_L^{\mathrm{sep},+}$$

form a fundamental system of open neighbourhoods of 0 *for this topology on $\mathbf{E}_L^{\mathrm{sep}}$.*

Proof.
This now is an immediate consequence of Lemma 3.3.10 and Lemma 3.3.12. □

3.4 Structure of Coefficient Rings (unramified case)

For this section, let $K|L$ be an unramified extension. Then it is a Galois extension and its Galois group is isomorphic to the Galois group of the respective residue class fields. It therefore is cyclic and generated by the lift of the q_L-Frobenius $x \mapsto x^{q_L}$. We will denote this lift by $\sigma_{K|L}$ and call it Frobenius on K. Recall also from Remark 3.1.2 that the groups Γ_L and Γ_K are isomorphic and for every $n \in \mathbb{N}$ the groups $\Gamma_{L_n|L}$ and Γ_{K_n} are isomorphic as well.

Remark 3.4.1.
We have $(H_L : H_K) = [K : L]$.

Proof.
Since $\Gamma_L \cong \Gamma_K$ (cf. Remark 3.1.2) we have $(H_L : H_K) = [K_\infty : L_\infty] = [K : L]$. □

Lemma 3.4.2.
We have $k_K \mathbf{E}_L = \mathbf{E}_{K|L}$.

Proof.
Since k_K is fixed by H_K it clearly is $k_K \mathbf{E}_L \subseteq \mathbf{E}_{K|L}$. Since $K|L$ is unramified we have $[K : L] = [k_K : k_L]$ and therefore

$$[k_K \mathbf{E}_L : \mathbf{E}_L] = [k_K : k_K \cap \mathbf{E}_L] = [k_K : k_L] = [K : L] = (H_L : H_K) = [\mathbf{E}_{K|L} : \mathbf{E}_L].$$

□

Lemma 3.4.3.
We have $\mathbf{A}_{K|L} = \mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L$ *and* $\mathbf{B}_{K|L} = K\mathbf{B}_L$.

Proof.
Since $K|L$ is unramified $\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L$ is unramified over $\mathbf{A}_L$ and since K is fixed by H_K we deduce $\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L \subseteq \mathbf{A}_{K|L}$. Since both are free $\mathbf{A}_L$-modules of rank $[K : L] = (H_L : H_K)$ they coincide.
The statement for the fields of fractions then follows immediately. □

In order to understand how the operations of Γ_K and the Frobenius look on $\mathbf{A}_{K|L}$ respectively $\mathbf{B}_{K|L}$ it now suffices to understand the corresponding operations on $\mathcal{O}_K$ respectively K. Note, that since $K|L$ is unramified, we clearly have $W(k_K)_L = \mathcal{O}_K$.

Lemma 3.4.4.
Let Fr *denote the (restriction of the)* q_L*-Frobenius on* k_K*. Then the automorphism* $\sigma_{K|L}$ *on* $\mathcal{O}_K$ *coincides with the restriction of* $W(\mathrm{Fr})_L$.

Proof.
Due to the functoriality of the Witt construction, $W(\mathrm{Fr})_L$ is an automorphism on $\mathcal{O}_K$ which fixes $\mathcal{O}_L$, it induces also an automorphism on K which fixes L and it's reduction modulo π_L is Fr. The first observation says, that the restriction of $W(\mathrm{Fr})_L$ is an element of $\mathrm{Gal}(K|L)$ and since $\mathrm{Gal}(K|L)$ and $\mathrm{Gal}(k_K|k_L)$ are isomorphic via $\sigma \mapsto \sigma \bmod \pi_L$, the second observation says that the restriction of $W(\mathrm{Fr})_L$ is a lift of Fr. Since this lift is unique we get the desired equality $W(\mathrm{Fr})_L = \sigma_{K|L}$ on K respectively $\mathcal{O}_K$. □

Before we give explicit descriptions of the operations on $\mathbf{A}_{K|L}$ we want to fix some notation.

Definition 3.4.5.
Let ϑ be an $\mathcal{O}_L$-linear endomorphism of $\mathcal{O}_K$ and $f \in \mathbf{A}_{K|L}$ we denote by f^ϑ the element, on which ϑ is applied to the coefficients of f, i.e. if $f(\omega_\phi) = \sum a_i \omega_\phi^i$ then

$$f^\vartheta(\omega_\phi) = \sum_{i \in \mathbb{N}_0} \vartheta(a_i)\omega_\phi^i.$$

Proposition 3.4.6.
Let $f = f(\omega_\phi) = \sum a_i \omega_\phi^i \in \mathbf{A}_{K|L}$ and $\gamma \in \Gamma_K$. We then have

$$\gamma \cdot f = \sum_{i \in \mathbb{Z}} a_i [\chi_{\mathrm{LT}}(\gamma)]_\phi(\omega_\phi^i).$$

For the Frobenius $\varphi_{K|L}$ we have

$$\varphi_{K|L}(f) = \sum_{i \in \mathbb{Z}} \sigma_{K|L}(a_i)[\pi_L]_\phi(\omega_\phi^i).$$

Together with the above Definition 3.4.5*, we then have the description*

$$\varphi_{K|L}(f(\omega_\phi)) = f^{\sigma_{K|L}}(\varphi_{K|L}(\omega_\phi)).$$

Proof.
This is an immediate consequence of Remark 3.1.2, Lemma 3.4.3 and Lemma 3.4.4. □

3.5 Structure of Coefficient Rings (general case)

Lemma 3.5.1.
Let k be a finite field of cardinality q and $E|k((X))$ be a finite, separable extension. Then it holds

$$E = \bigoplus_{i=0}^{q-1} X^i E^q.$$

Proof. This is [Kup15, Lemma 1.39, p. 21]. □

Proposition 3.5.2.

$$\mathbf{A} = \bigoplus_{i=0}^{q_L - 1} \mathrm{Fr}(\mathbf{A})\omega_\phi^i.$$

Proof. This is [Kup15, Proposition 1.41, p. 21–22]. □

Corollary 3.5.3.

$$\mathbf{A}_{K|L} = \bigoplus_{i=0}^{q_L-1} \varphi_{K|L}(\mathbf{A}_{K|L})\omega_\phi^i.$$

Proposition 3.5.4.
Let $B|\mathbf{B}_L$ be a finite, unramified extension and $A \subseteq B$ be the integral closure of $\mathbf{A}_L$. Then there exists a finite, unramified extension $E|L$ and an element $\nu_\phi \in W(\mathbf{E}_L^{\mathrm{sep}})_L$ with $\nu_\phi^j = \omega_\phi$ for some $j > 0$ such that

$$A \cong \varprojlim_n \mathcal{O}_E/\pi_L^n\mathcal{O}_E((\nu_\phi)).$$

Proof.
Let κ be the residue class field of A and recall that the residue class field of $\mathbf{A}_L$ is $\mathbf{E}_L = k_L((\omega))$. Since $B|\mathbf{B}_L$ is unramified, we then have

$$[B : \mathbf{B}_L] = [\kappa : k_L((\omega))].$$

Since $\kappa|k_L((\omega))$ is finite and separable ($B|\mathbf{B}_L$ is unramified), we deduce from Lemma 3.3.8 that $\kappa \cong k((\nu))$ for some finite extension $k|k_L$ and $\nu \in \mathbf{E}_L^{\mathrm{sep}}$ with $\nu^j = \omega$ for some $j > 0$. But then there exists a unique finite and unramified extension $E|L$ with $k_E = k$. In particular, we have $j[E : L] = [B : \mathbf{B}_L]$. Furthermore, since $\mathbf{B}_L$ is a complete discrete valuation field, and $B|\mathbf{B}_L$ is a finite extension, the henselian lemma in the sense of [Neu07, II §4, (4.6) Henselsches Lemma, p. 135–136] holds true and therefore we can find a $\nu_\phi \in B \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ which is a root of the polynomial $X^j - \omega_\phi$ and for which we have

$$\nu_\phi \bmod \pi_L = \nu.$$

Since $X^j - \omega_\phi$ is irreducible over $\mathbf{B}_L$ and $E|L$ is unramified, we deduce

$$\begin{aligned}
[E\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L] &= [E\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L(\nu_\phi)] \cdot [\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L] \\
&= [E : E \cap \mathbf{B}_L(\nu_\phi)] \cdot j \\
&= [E : L] \cdot j
\end{aligned}$$

and therefore

$$B = E\mathbf{B}_L(\nu_\phi).$$

In particular, we have

$$B = \left\{ \sum_{i\in\mathbb{Z}} a_i \nu_\phi^i \;\middle|\; a_i \in E,\ \lim_{i\to-\infty} a_i = 0 \text{ and it exists } n \in \mathbb{N} \right.$$
$$\left. \text{such that } \pi_L^n a_i \in \mathcal{O}_E \text{ for all } i \in \mathbb{Z} \right\}$$

Then A consists of those elements of B with π_L-adic absolute value ≤ 1. Since this absolute value is nonarchimedean, these are exactly those elements $\sum_{i\in\mathbb{Z}} a_i\nu_\phi^i \in B$ with $a_i \in \mathcal{O}_E$ for all $i \in \mathbb{Z}$, i.e.

$$A \cong \varprojlim_n \mathcal{O}_E/\pi_L^n \mathcal{O}_E((\nu_\phi)).$$

□

3.6 $(\varphi_{K|L}, \Gamma_K)$-MODULES AND GALOIS REPRESENTATIONS

Before we give the definition of $(\varphi_{K|L}, \Gamma_K)$-modules, we want to recall some useful tools. If not otherwise stated, all continuity statements refer to the corresponding weak topology.

Definition 3.6.1.
Let M be an $\mathbf{A}_{K|L}$-module. We regard M as a left-$\mathbf{A}_{K|L}$-module and $\mathbf{A}_{K|L}$ itself as a right-$\mathbf{A}_{K|L}$-module via $\varphi_{K|L}$. For the tensor product in this situation we write $\mathbf{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} M$, which is per definition an abelian group, but since $\mathbf{A}_{K|L}$ is also a left- $\mathbf{A}_{K|L}$-module (with the standard multiplication) this tensor product is also a (left)-$\mathbf{A}_{K|L}$-module.

Remark 3.6.2.
As a set $\mathbf{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} M$ is equal to the standard tensor product $\mathbf{A}_{K|L} \otimes_{\mathbf{A}_{K|L}} M$, but since we regard $\mathbf{A}_{K|L}$ as right-$\mathbf{A}_{K|L}$-module via $\varphi_{K|L}$, we have the relation

$$x \otimes am = x\varphi_{K|L}(a) \otimes m,$$

for all $x, a \in \mathbf{A}_{K|L}$ and $m \in M$. Note also that we have

$$a(x \otimes m) = (ax) \otimes m$$

for all $a, x \in \mathbf{A}_{K|L}$ and $m \in M$.

Lemma 3.6.3.
The functor

$$\mathrm{Mod}(\mathbf{A}_{K|L}) \to \mathrm{Mod}(\mathbf{A}_{K|L}), M \mapsto \mathbf{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} M$$

is exact.

Proof.
Since $\mathbf{A}_{K|L}$ is a discrete valuation ring and $\mathbf{A}_{K|L}$ is free as (right-)$\mathbf{A}_{K|L}$-module via $\varphi_{K|L}$ (cf. Proposition 3.5.3), this is [Bou61, Proposition 3, p. 29]. □

Definition 3.6.4.
Let M be a finitely generated $\mathbf{A}_{K|L}$-module equipped with a $\varphi_{K|L}$-linear endomorphism φ_M. Then φ_M^{lin} denotes the homomorphism

$$\begin{aligned} \varphi_M^{\mathrm{lin}} \colon \mathbf{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} M &\longrightarrow M \\ f \otimes m &\longmapsto f\varphi_M(m). \end{aligned}$$

Definition 3.6.5.
A finitely generated $\mathbf{A}_{K|L}$-module M is called $(\varphi_{K|L}, \Gamma_K)$**-module** if it is equipped with a $\varphi_{K|L}$-linear endomorphism φ_M and a continuous, semilinear action from Γ_K, which commutes with the endomorphism φ_M.
A $(\varphi_{K|L}, \Gamma_K)$-module is called **étale** if the homomorphism φ_M^{lin} is bijective.
A morphism of $(\varphi_{K|L}, \Gamma_K)$-modules $f \colon M \to N$ is an $\mathbf{A}_{K|L}$-module homomorphism, which respects the actions from Γ_K and the endomorphisms φ_M and φ_N.
We will denote the category of étale $(\varphi_{K|L}, \Gamma_K)$-modules by $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$.

Definition 3.6.6.
We denote the category of finitely generated $\mathcal{O}_L$-modules together with a continuous $\mathcal{O}_L$-linear action from G_K, the so called G_K-representations of $\mathcal{O}_L$, by $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$.

We now will define two functors between $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ and $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ (one in each direction) from which we then in the following section will prove that they define an equivalence of these categories.

Definition 3.6.7.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$. We then define

$$\mathcal{V}_{\mathcal{K}|L}(M) := (\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M)^{\mathrm{Fr}\otimes\varphi_M = 1}.$$

Remark 3.6.8.
At the moment, $\mathcal{V}_{K|L}$ gives a functor from $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ to the category of $\mathcal{O}_L$-modules with a group action from G_K. If we want to see that it is a functor with image in $\mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ we have to show that for $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ it holds:

1. *$\mathcal{V}_{K|L}(M)$ is finitely generated as $\mathcal{O}_L$-module.*
2. *The G_K-action is continuous.*

Before we go into the prove of this, we want to define the functor in the opposite direction and explain what we have to prove in order to see that it is well defined.

Definition 3.6.9.
Let $V \in \mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$. We then define

$$\mathcal{M}_{K|L}(V) := (\mathbf{A} \otimes_{\mathcal{O}_L} V)^{H_K}.$$

Remark 3.6.10.
As before, at the moment, $\mathcal{M}_{K|L}$ defines a functor from $\mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ to the category of $\mathscr{A}_{K|L}$-modules which have a group action from Γ_K and an endomorphism induced from Fr which commutes with the action from Γ_K. In order to see that $\mathcal{M}_{K|L}$ has image in $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ we have to show that for $V \in \mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ it holds:

1. *$\mathcal{M}_{K|L}(V)$ is finitely generated as $\mathscr{A}_{K|L}$-module.*
2. *The endomorphism induced from Fr is continuous.*
3. *The Γ_K-action is continuous.*
4. *$\varphi^{\text{lin}}_{\mathcal{M}_{K|L}(V)}$ is an isomorphism.*

The proof, that the categories $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ and $\mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ are equivalent, will be the content of the next three sections.
In the first section, we give the general idea of the proof and explain what we exactly have to prove. In the following section we will give a proof in characteristic p and in the last section, we deduce from this the general equivalence. Our exposition in these sections follows [Sch17, Chapter 3, p. 110–135] and explains how the results from there transform to our situation.

3.7 The strategy for the equivalence $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K) \cong \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$

In this section we want to prove as many of the conditions of Remark 3.6.8 and Remark 3.6.10 as possible in the general case. For this we introduce two comparison homomorphisms, which will give us some nice results if we can prove that they are isomorphisms. The proof of the bijectivity will be the part of the following sections. Additionally, they will lead us to the desired equivalence.

Lemma 3.7.1.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. Then

$$\begin{aligned} \varphi^{\mathrm{lin}}_{\mathcal{M}_{K|L}(V)} \colon \mathscr{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathbf{A}_{K|L}} \mathcal{M}_{K|L}(V) &\longrightarrow \mathcal{M}_{K|L}(V), \\ f \otimes m &\longmapsto f \otimes \varphi_{\mathcal{M}_{K|L}(V)}(m) \end{aligned}$$

is an isomorphism.

Proof.
Because Corollary 3.5.3 and Lemma 3.6.3 say that $\mathbf{A}_{K|L}$ has the same properties as $\mathbf{A}_L$ which are needed for [Sch17, Lemma 3.1.7, p. 116–117], the proof is the same as the one of loc. cit. □

Definition 3.7.2.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. We then define

$$\begin{aligned} \mathrm{ad}_V \colon \mathbf{A} \otimes_{\mathbf{A}_{K|L}} \mathcal{M}_{K|L}(V) &\longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V \\ a \otimes m &\longmapsto am. \end{aligned}$$

Remark 3.7.3.
For $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ the map ad_V is a homomorphism of $\mathbf{A}$-modules, it is compatible with the diagonal G_K-actions on both sides and it satisfies

$$\mathrm{ad}_V \circ (\mathrm{Fr} \otimes \varphi_{\mathcal{M}_{K|L}(V)}) = (\mathrm{Fr} \otimes \mathrm{id}) \circ \mathrm{ad}_V.$$

Definition 3.7.4.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$. We then define

$$\begin{aligned} \mathrm{ad}_M \colon \mathbf{A} \otimes_{\mathcal{O}_L} \mathcal{V}_{K|L}(M) &\longrightarrow \mathbf{A} \otimes_{\mathbf{A}_{K|L}} M \\ a \otimes v &\longmapsto av. \end{aligned}$$

Remark 3.7.5.
For $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ the map ad_M is a homomorphism of $\mathbf{A}$-modules, it is compatible with the diagonal G_K-action on both sides and it satisfies

$$\mathrm{ad}_M \circ (\mathrm{Fr} \otimes \mathrm{id}) = (\mathrm{Fr} \otimes \varphi_M) \circ \mathrm{ad}_M$$

Lemma 3.7.6.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. Then the diagonal action from G_K on $\mathbf{A} \otimes_{\mathcal{O}_L} V$ is continuous for the tensor product topology, where $\mathbf{A}$ carries its weak topology.

Proof.
This is literally the same as the proof of [Sch17, Lemma 3.1.10, p. 119–120]. □

Proposition 3.7.7.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and assume that $\mathcal{M}_{K|L}(V)$ is finitely generated as $\mathbf{A}_{K|L}$-module as well as ad_V is an isomorphism.
Then the induced Γ_K-action on $\mathcal{M}_{K|L}(V)$ and the endomorphism $\varphi_{\mathcal{M}_{K|L}(V)}$ induced from Fr *are continuous for the weak topology and $\mathcal{M}_{K|L}(V)$ has the same elementary divisors as V. This means, that if we have $\mathcal{M}_{K|L}(V) \cong \oplus_{i=1}^s \mathbf{A}_{K|L}/\pi_L^{m_i}\mathbf{A}_{K|L}$ with $1 \leq m_1 \leq \cdots \leq m_s \leq \infty$ as $\mathbf{A}_{K|L}$-modules by the main theorem for finitely generated modules over principal ideal domains (cf.* [Bos09, Korollar 7, p. 80]*), then we also have $V \cong \oplus_{i=1}^s \mathcal{O}_L/\pi_L^{m_i}\mathcal{O}_L$ as $\mathcal{O}_L$-modules.*

Proof.
Lemma 3.7.6 induces that

$$G_K \times \mathcal{M}_{K|L}(V) \longrightarrow G_K \times \mathbf{A} \otimes_{\mathcal{O}_L} V \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V$$

is continuous and since this map has image in $\mathcal{M}_{K|L}(V)$ and reduces to the action of Γ_K, since $\mathcal{M}_{K|L}(V)$ is, by definition, H_K-invariant, Lemma 3.2.5 says that the Γ_K-action on $\mathcal{M}_{K|L}(V)$ is continuous for the topology induced from the weak topology of $\mathbf{A} \otimes_{\mathcal{O}_L} V$. Similarly, since

$$\mathcal{M}_{K|L}(V) \hookrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}_V} \mathbf{A} \otimes_{\mathcal{O}_L} V$$

is continuous with image in $\mathcal{M}_{K|L}(V)$, Lemma 3.2.5 says that $\varphi_{\mathcal{M}_{K|L}(V)}$ is continuous on $\mathcal{M}_{K|L}(V)$ for the topology induced from the weak topology of $\mathbf{A} \otimes_{\mathcal{O}_L} V$. One checks that this topology coincides with the weak topology on $\mathcal{M}_{K|L}(V)$ and the statement on the elementary divisors as in [Sch17, Proposition 3.1.12, p. 122] where we make us of Proposition 3.3.6 instead of [Sch17, Lemma 3.1.8, p. 118–119]. □

Remark 3.7.8.
The above Proposition 3.7.7 together with Lemma 3.7.1 says that in order to check the conditions from Remark 3.6.10 it suffices to check that for $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ it holds:

1. *$\mathcal{M}_{K|L}(V)$ is finitely generated as $\mathbf{A}_{K|L}$-module.*
2. *ad_V is bijective.*

Lemma 3.7.9.
Let $M \in \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{A}_{K|L})$. Then the diagonal action from G_K on $\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M$ is continuous for the tensor product topology of the weak topologies on both sides.

Proof.
Since G_K acts continuously on both, $\mathbf{A}$ and M, this is exactly the same as [Sch17, Lemma 3.1.11, p. 120–122] (which makes only use of these properties). □

Proposition 3.7.10.
Let $M \in \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{A}_{K|L})$ and assume that $\mathcal{V}_{K|L}(M)$ is finitely generated over $\mathcal{O}_L$ as well as ad_M is an isomorphism.
Then the diagonal action from G_K on $\mathcal{V}_{K|L}(M)$ is continuous for the π_L-adic topology and $\mathcal{V}_{K|L}(M)$ has the same elementary divisors as M. This is to be understood is in the above Proposition 3.7.7:
If $\mathcal{V}_{K|L}(M) \cong \oplus_{i=1}^{s} \mathcal{O}_L/\pi_L^{m_i}\mathcal{O}_L$ with $1 \leq m_1 \leq \cdots \leq m_s \leq \infty$ as $\mathcal{O}_L$-modules, then $M \cong \oplus_{i=1}^{s} \mathbf{A}_{K|L}/\pi_L^{m_i}\mathbf{A}_{K|L}$ as $\mathbf{A}_{K|L}$-modules.

Proof.
Lemma 3.7.9 induces that

$$G_K \times \mathcal{V}_{K|L}(M) \longrightarrow G_K \times (\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M) \longrightarrow \mathbf{A} \otimes_{\mathbf{A}_{K|L}} M$$

is continuous and since this map has image in $\mathcal{V}_{K|L}(M)$, Lemma 3.2.5 says that the G_K-action on $\mathcal{V}_{K|L}(M)$ is continuous for the topology on $\mathcal{V}_{K|L}(M)$ which is induced from the weak topology on $\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M$. One checks that this topology coincides with the weak topology and the statement on the elementary divisors as in [Sch17, Proposition 3.1.13, p. 122–123]. □

Remark 3.7.11.
The above Proposition 3.7.10 says that in order to check the conditions from Remark 3.6.8 it suffices to check that for $M \in \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{A}_{K|L})$ it holds:

1. *$\mathcal{V}_{K|L}(M)$ is finitely generated as $\mathcal{O}_L$-module.*
2. *ad_M is bijective.*

3.8 THE EQUIVALENCE $\mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K) \cong \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L})$

In this section, we want to explain, why the categories in question are equivalent if the corresponding objects are annihilated by π_L. This is nearly similar to [Sch17, Section 3.2, p. 123–129] since the hard facts proved there are in such a generality, that they also cover our situation. Nevertheless, we will write down the statements in the relative situation, we have chosen.

Remark 3.8.1.
If $M \in \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{A}_{K|L})$ is annihilated by π_L, then M is clearly a finite dimensional $\mathbf{E}_{K|L}$-vector space and its weak topology coincides with the natural topology as $\mathbf{E}_{K|L}$-vector space. We denote the corresponding category by $\mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L})$. For the functor $\mathcal{V}_{K|L}$ we then obtain

$$\mathcal{V}_{K|L}(M) = \left(\mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_{K|L}} M\right)^{\mathrm{Fr}\otimes\varphi_M=1}.$$

Analogous, if $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ is annihilated by π_L, then V is a finite dimensional k_L-vector space and its G_K-action is continuous for the discrete topology. The corresponding category will be denoted by $\mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K)$. For the functor $\mathcal{M}_{K|L}$ we then obtain

$$\mathcal{M}_{K|L}(V) = \left(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V\right)^{H_K}.$$

Lemma 3.8.2.
Let $V \in \mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K)$ and let $v_1, \ldots, v_k$ be a k_L-basis of V. Then there exists a normal open subgroup $N \lhd G_K$ such that $\sigma(v_i) = v_i$ for all $\sigma \in N$ and $1 \le i \le k$.

Proof.
For $1 \le i \le k$ set

$$N_i := \ker(G_K \to V,\ \sigma \mapsto \sigma(v_i) - v_i).$$

Since the G_K-action on V is continuous and V carries the discrete topology, each N_i is open and normal in G_K. Then take $N := \cap_i N_i$ □

The following Proposition 3.8.3 is exactly [Sch17, Proposition 3.2.1, p. 123–124], where the above Lemma 3.8.2 explains one small step in detail.

Proposition 3.8.3.
Let $V \in \mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K)$. Then it holds:

1. *The $\mathbf{E}_L^{\mathrm{sep}}$-vector space $\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V$ has a basis consisting of elements, which are fixed by H_K.*

2. $\mathcal{M}_{K|L}(V)$ *is a finitely generated* $\mathbf{E}_{K|L}$*-vector space.*

3. ad_V *is bijective.*

Proof.
Although this is, as stated before, exactly [Sch17, Proposition 3.2.1, p. 123–124], we want to state the (very short) proves of 2. and 3. For this, let $v_1, \dots, v_k$ be an $\mathbf{E}_L^{\mathrm{sep}}$-basis of $\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V$ which is fixed by H_K.

2. $$\begin{aligned} \mathcal{M}_{K|L}(V) &= \left(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V\right)^{H_K} \\ &= \left(\mathbf{E}_L^{\mathrm{sep}} v_1 + \dots + \mathbf{E}_L^{\mathrm{sep}} v_k\right)^{H_K} \\ &= \left(\mathbf{E}_L^{\mathrm{sep}}\right)^{H_K} v_1 + \dots + \left(\mathbf{E}_L^{\mathrm{sep}}\right)^{H_K} v_k \\ &= \mathbf{E}_{K|L} v_1 + \dots + \mathbf{E}_{K|L} v_k. \end{aligned}$$

3. $$\begin{aligned} \mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_{K|L}} \mathcal{M}_{K|L}(V) &\overset{1}{=} \mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_{K|L}} \left(\mathbf{E}_{K|L} v_1 + \dots + \mathbf{E}_{K|L} v_k\right) \\ &= \mathbf{E}_L^{\mathrm{sep}} v_1 + \dots + \mathbf{E}_L^{\mathrm{sep}} v_k \\ &= \mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V. \end{aligned}$$

□

Proposition 3.8.4.
Let $M \in \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L})$*. Then* $\mathcal{V}_{K|L}(M)$ *is a finite dimensional* k_L*-vector space and* ad_M *is bijective.*

Proof.
This is [Sch17, Proposition 3.2.4, p. 126–128] with $F = \mathbf{E}_L^{\mathrm{sep}}$, $W = \mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_{K|L}} M$ and $f = \mathrm{Fr} \otimes \varphi_M$. □

Theorem 3.8.5.
The categories $\mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K)$ *and* $\mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L})$ *are equivalent. The equivalence is given by the quasi invers functors*

$$\begin{aligned} \mathcal{M}_{K|L} \colon \mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K) &\longrightarrow \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L}) \\ V &\longmapsto \left(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V\right)^{H_K} \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_{K|L} \colon \mathbf{Mod}_{\varphi,\Gamma}^{\text{ét}}(\mathbf{E}_{K|L}) &\longrightarrow \mathbf{Rep}_{k_L}^{(\mathrm{fg})}(G_K) \\ M &\longmapsto \left(\mathbf{E}_L^{\mathrm{sep}} \otimes_{\mathbf{E}_{K|L}} M\right)^{\mathrm{Fr} \otimes \varphi_M = 1}. \end{aligned}$$

Proof.
This is [Sch17, Corollary 3.2.3, p. 126], [Sch17, Corollary 3.2.6, p. 129] and [Sch17, Corollary 3.2.7, p. 129]. For the completeness of this section, we want to calculate the quasi inversion of $\mathcal{M}_{K|L}$ and $\mathcal{V}_{K|L}$.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{E}_{K|L})$. Then:

$$\begin{aligned}\mathcal{M}_{K|L}(\mathcal{V}_{K|L}(M)) &= \left(\mathbf{E}_L^{\text{sep}} \otimes_{k_L} \mathcal{V}_{K|L}(M)\right)^{H_K} \\ &\overset{\text{ad}_M}{\cong} \left(\mathbf{E}_L^{\text{sep}} \otimes_{\mathbf{E}_{K|L}} M\right)^{H_K} \\ &= \left(\mathbf{E}_L^{\text{sep}}\right)^{H_K} \otimes_{\mathbf{E}_{K|L}} M \\ &= \mathbf{E}_{K|L} \otimes_{\mathbf{E}_{K|L}} M = M.\end{aligned}$$

Let $V \in \mathbf{Rep}^{(\text{fg})}_{k_L}(G_K)$. Then:

$$\begin{aligned}\mathcal{V}_{K|L}(\mathcal{M}_{K|L}(V)) &= \left(\mathbf{E}_L^{\text{sep}} \otimes_{\mathbf{E}_{K|L}} \mathcal{M}_{K|L}(V)\right)^{\text{Fr}\otimes\varphi_{\mathcal{M}_{K|L}(V)}=1} \\ &\overset{\text{ad}_V}{\cong} \left(\mathbf{E}_L^{\text{sep}} \otimes_{k_L} V\right)^{\text{Fr}\otimes\text{id}_V=1} \\ &= \left(\mathbf{E}_L^{\text{sep}}\right)^{\text{Fr}=1} \otimes_{k_L} V \\ &= k_L \otimes_{k_L} V = V.\end{aligned}$$

□

3.9 The equivalence $\mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K) \cong \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$

Since the new key inputs in [Sch17, section 3.3, p. 129–135] (namely [Sch17, Remark 3.3.1, p. 129] and [Sch17, Remark 3.3.5, p. 133]) are formulated in a generality in which also our situation fits, the proof of the general equivalence is completely analogous to [Sch17, section 3.3, p. 129–135]. Therefore, we only want to state it here.

Theorem 3.9.1.
The categories $\mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K)$ and $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ are equivalent to each other. The equivalence is given by the quasi invers functors

$$\begin{aligned}\mathcal{M}_{K|L}\colon \mathbf{Rep}^{(\text{fg})}_{\mathcal{O}_L}(G_K) &\longrightarrow \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L}) \\ V &\longmapsto (\mathbf{A} \otimes_{\mathcal{O}_L} V)^{H_K}\end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_{K|L} \colon \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L}) &\longrightarrow \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K) \\ M &\longmapsto \Big(\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M\Big)^{\mathrm{Fr}\otimes\varphi_M = 1}. \end{aligned}$$

Chapter 4

Iwasawa Cohomology and an explicit Reciprocity Law

In this chapter we will generalize [SV15, Theorem 6.2, p. 32] to finite, unramified extensions. So, we keep the notations from the previous chapters but we assume additionally that $K|L$ is an **unramified** extension and let $d_{K|L} := [K : L]$ denote the degree of the extension. This assumption leads to some simplifications of the involved structures, which we will discuss before we turn to the results of [SV15].

4.1 Coleman Power Series

Now we head towards [SV15]. First we want to recall some notations from there. For this, we are very close to [SV15, Notation, p. 3] and we keep most of the notations from there to avoid confusion and to simplify comparisons. As at their beginning, we make use of [Col79] and therefore work first with power series rings, rings of formal Laurent series and completions of these rings. Due to Lemma 3.4.3 it makes sense to work with the same coordinate over both, L and K which we will denote by Z as in [SV15, p. 3]. Recall from Chapter 3 that we fixed a Lubin-Tate power series $\phi \in \mathcal{O}_L[\![Z]\!]$ and an associated Lubin-Tate formal group $\mathcal{G}_\phi \in \mathcal{O}_L[\![X, Y]\!]$. As in [SV15, p. 3] we write $X +_{\mathcal{G}_\phi} Y$ instead of $\mathcal{G}_\phi(X, Y)$. In our opinion, this leads to a clearer presentation.

Remark 4.1.1.
Since $\mathcal{G}_\phi \equiv X + Y \mod \deg 2$*, the power series* $\frac{\partial(\mathcal{G}_\phi(X,Y))}{\partial Y}\big|_{(X,Y)=(0,Z)}$ *has no constant term and therefore has an inverse in* $\mathcal{O}_L[\![Z]\!]$*.*

Definition 4.1.2.
Let $g_{\mathrm{LT}} \in \mathcal{O}_L[\![Z]\!]$ denote the inverse of $\frac{\partial(\mathcal{G}_\phi(X,Y))}{\partial Y}\big|_{(X,Y)=(0,Z)}$ (cf. Remark 4.1.1). Let furthermore $\log_{\mathrm{LT}}(Z) = Z + \cdots \in L[\![Z]\!]$ denote the unique formal power series whose formal derivative is g_{LT}.

Remark 4.1.3.
$g_{\mathrm{LT}}(Z)\mathrm{d}Z$ is the unique invariant differential form on $\mathcal{G}_\phi$.
$\log_{\mathrm{LT}}$ is the logarithm of $\mathcal{G}_\phi$. In particular, we have $g_{\mathrm{LT}}(Z)\mathrm{d}Z = \mathrm{d}\log_{\mathrm{LT}}(Z)$.

Proof.
The first statement is [Haz78, §5.8], the second [Lan78, p. 8.6]. □

Definition 4.1.4.
By ∂_{inv} we denote the invariant derivation corresponding to $\mathrm{d}\log_{\mathrm{LT}}$, i.e. for $f \in \mathcal{O}_L[\![Z]\!]$ we have

$$\mathrm{d}f = \partial_{\mathrm{inv}}(f)\mathrm{d}\log_{\mathrm{LT}} .$$

Remark 4.1.5.
For $f \in \mathcal{O}_L[\![Z]\!]$ we have

$$\partial_{\mathrm{inv}}(f) = \frac{f'}{g_{\mathrm{LT}}}.$$

Proof.
We have (cf. [SV15, p. 3])

$$f'\mathrm{d}Z = \mathrm{d}f = \partial_{\mathrm{inv}}(f)\mathrm{d}\log_{\mathrm{LT}} = \partial_{\mathrm{inv}}(f)g_{\mathrm{LT}}\mathrm{d}Z$$

which immediately implies the claim. □

Remark 4.1.6.
∂_{inv} clearly is $\mathcal{O}_L$-linear and therefore it is continuous for the π_L-adic topology on $\mathcal{O}_L[\![Z]\!]$.

Remark 4.1.7.
By the same formula as above (cf. Remark 4.1.5), we expand ∂_{inv} to $\mathcal{O}_K[\![Z]\!]$, i.e. if $f \in \mathcal{O}_K[\![Z]\!]$ we set

$$\partial_{\mathrm{inv}}(f) = \frac{f'}{g_{\mathrm{LT}}}.$$

Then clearly ∂_{inv} is $\mathcal{O}_K$-linear and therefore it is continuous for the π_L-adic topology on $\mathcal{O}_K[\![Z]\!]$.

Remark 4.1.8.
For $a \in \mathcal{O}_L$ we have

$$\log_{\mathrm{LT}}([a](Z)) = a \cdot \log_{\mathrm{LT}}(Z)$$
$$a \cdot g_{\mathrm{LT}}(Z) = g_{\mathrm{LT}}([a](Z)) \cdot [a]'(Z).$$

Proof.
This is [Lan78, 8.6, Lemma 2]. □

As before (cf. Definition 3.4.5), if ϑ is an $\mathcal{O}_L$-linear endomorphism of $\mathcal{O}_K$ and $f(Z) \in \mathcal{O}_K[\![Z]\!]$ we denote by $f^{\vartheta}(Z)$ the power series, on which ϑ is applied to its coefficients, i.e. if $f(Z) = \sum a_i Z^i$ then

$$f^{\vartheta}(Z) = \sum_{i \in \mathbb{N}_0} \vartheta(a_i) Z^i.$$

Recall that on $\mathcal{O}_L[\![Z]\!]$ we have the Frobenius endomorphism (cf. [SV15, p. 3], [Col79, p. 97], or here)

$$\varphi_L : \mathcal{O}_L[\![Z]\!] \to \mathcal{O}_L[\![Z]\!],\ f(Z) \mapsto f([\pi_L]_\phi(Z)).$$

Together with Lemma 3.4.4 the corresponding Frobenius endomorphism on $\mathcal{O}_K[\![Z]\!]$ then is

$$\varphi_{K|L} : \mathcal{O}_K[\![Z]\!] \to \mathcal{O}_K[\![Z]\!],\ f(Z) \mapsto f^{\sigma_{K|L}}([\pi_L]_\phi(Z)).$$

Remark 4.1.9.
This endomorphism $\varphi_{K|L}$ on $\mathcal{O}_K[\![Z]\!]$ is a bit different to the one in [Col79, p. 97] *although Coleman also works with a finite unramified extension over his fixed base field (in contrast to* [SV15, p. 3]*). The reason of this difference is, that the above endomorphism translates to the endomorphism on the coefficient ring of Lubin-Tate $(\varphi_{K|L}, \Gamma_K)$-modules (cf.* Lemma 3.4.4*) in which we are interested later on. We now have to check, that the results of* [Col79] *translates to our situation.*
We will also make use of the endomorphism defined by Coleman and in order to keep the notation from [SV15]*, we will denote by φ_L the following endomorphism of $\mathcal{O}_K[\![Z]\!]$:*

$$\varphi_L : \mathcal{O}_K[\![Z]\!] \to \mathcal{O}_K[\![Z]\!],\ f(Z) \mapsto f([\pi_L]_\phi(Z)).$$

Note that at [SV15, p. 3] *φ_L is defined on $\mathcal{O}_L[\![Z]\!]$, but* [Col79, p. 97] *defines it in this way. So, our φ_L induces the φ_L of* [SV15] *by restriction.*

We now want to characterize the image of $\varphi_{K|L}$. First, we recall the description of the image of φ_L.

Lemma 4.1.10.
It is

$$\mathrm{im}(\varphi_L) = \{f \in \mathcal{O}_K[\![Z]\!] |\ f(Z) = f(a +_{\mathcal{G}_\phi} Z) \text{ for all } a \in \mathcal{G}_\phi[\pi_L]\}.$$

Proof.
See [Col79, Lemma 3, p. 97]. □

Remark 4.1.11.
Let $\vartheta \in \mathrm{Gal}(K|L)$ *and* $f \in \mathcal{O}_K[\![Z]\!]$

1. *Because* $[\pi_L]_\phi(Z) \in \mathcal{O}_L[\![Z]\!]$ *we have* $[\pi_L]_\phi(Z) = [\pi_L]_\phi^\vartheta(Z)$ *and therefore*

$$(f^\vartheta \circ [\pi_L]_\phi)(Z) = (f^\vartheta \circ [\pi_L]_\phi^\vartheta)(Z) = (f \circ [\pi_L]_\phi)^\vartheta(Z).$$

2. *With this, we accomplish* $\varphi_{K|L}(f^\vartheta) = (\varphi_{K|L}(f))^\vartheta$*, since*

$$\begin{aligned}\varphi_{K|L}(f^\vartheta) &= (f^\vartheta)^{\varphi_{K|L}}([\pi_L]_\phi(Z)) = (f^{\sigma_{K|L}})^\vartheta([\pi_L]_\phi(Z)) \\ &= (f^{\sigma_{K|L}}([\pi_L]_\phi(Z)))^\vartheta = (\varphi_{K|L}(f))^\vartheta,\end{aligned}$$

where the second equality holds, because ϑ *and* $\sigma_{K|L}$ *are elements of* $\mathrm{Gal}(K|L)$*, which is cyclic.*

Corollary 4.1.12.
In particular, we have

$$\varphi_{K|L} = \sigma_{K|L} \circ \varphi_L = \varphi_L \circ \sigma_{K|L}.$$

Proof.
This is exactly the first statement of Remark 4.1.11. □

Proposition 4.1.13.
We have

$$\mathrm{im}(\varphi_{K|L}) = \{f \in \mathcal{O}_K[\![Z]\!] |\ f(Z) = f(a +_{\mathcal{G}_\phi} Z) \text{ for all } a \in \mathcal{G}_\phi[\pi_L]\}.$$

Proof.
Since $\sigma_{K|L}$ is an isomorphism on $\mathcal{O}_K$ and therefore also on $\mathcal{O}_K[\![Z]\!]$ and because of

$$\varphi_{K|L} = \sigma_{K|L} \circ \varphi_L$$

(cf. Corollary 4.1.12) this is an immediate consequence of Lemma 4.1.10. □

Coleman then continues in [Col79, Thm. 4, Cor. 5, p. 98], to prove the existence of a unique $\mathcal{O}_L$-linear endomorphism $\widetilde{\psi_{\mathrm{Col}}} \colon \mathcal{O}_K[\![Z]\!] \to \mathcal{O}_K[\![Z]\!]$ such that for $f \in \mathcal{O}_K[\![Z]\!]$

$$(\varphi_L \circ \widetilde{\psi_{\mathrm{Col}}})(f(Z)) = \sum_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z)$$

and in [Col79, Thm. 11, p. 102] of a unique multiplicative map $\widetilde{\mathcal{N}} \colon \mathcal{O}_K[\![Z]\!] \to \mathcal{O}_K[\![Z]\!]$ such that for $f \in \mathcal{O}_K[\![Z]\!]$

$$(\varphi_L \circ \widetilde{\mathcal{N}})(f(Z)) = \prod_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z).$$

The map $\widetilde{\mathcal{N}}$ is called **norm operator**.

Remark 4.1.14.
For all $f \in \mathcal{O}_K[\![Z]\!]$ and $\vartheta \in \mathrm{Gal}(K|L)$ we have

1. $\widetilde{\psi_{\mathrm{Col}}}(f^\vartheta(Z)) = (\widetilde{\psi_{\mathrm{Col}}}(f))^\vartheta(Z)$.
2. $\widetilde{\mathcal{N}}(f^\vartheta(Z)) = (\widetilde{\mathcal{N}}(f))^\vartheta(Z)$.

Proof.
Because φ_L is injective, it suffices the check the equations after applying φ_L.

1. $$\begin{aligned}(\varphi_L \circ \widetilde{\psi_{\mathrm{Col}}})(f^\vartheta(Z)) &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} (f^\vartheta)(a +_{\mathcal{G}_\phi} Z) \\ &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} (f)^\vartheta(a +_{\mathcal{G}_\phi} Z) \\ &= \left(\sum_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z) \right)^\vartheta \\ &= \left((\varphi_L \circ \widetilde{\psi_{\mathrm{Col}}})(f) \right)^\vartheta (Z) \\ &= \varphi_L(\widetilde{\psi_{\mathrm{Col}}}(f)^\vartheta)(Z).\end{aligned}$$

   The third equality holds true since $\mathcal{G}_\phi$ has coefficients in $\mathcal{O}_L$.

2. $$\begin{aligned}(\varphi_L \circ \widetilde{\mathcal{N}})(f^{\vartheta}(Z)) &= \prod_{a \in \mathcal{G}_\phi[\pi_L]} (f^{\vartheta})(a +_{\mathcal{G}_\phi} Z)\\ &= \prod_{a \in \mathcal{G}_\phi[\pi_L]} (f)^{\vartheta}(a +_{\mathcal{G}_\phi} Z)\\ &= \left(\prod_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z)\right)^{\vartheta}\\ &= \left((\varphi_L \circ \widetilde{\mathcal{N}})(f)\right)^{\vartheta}(Z)\\ &= \varphi_L(\widetilde{\mathcal{N}}(f)^{\vartheta})(Z).\end{aligned}$$

As above, the third equality holds true since $\mathcal{G}_\phi$ has coefficients in $\mathcal{O}_L$.

□

Remark 4.1.15.
Let $f \in \mathcal{O}_K[\![Z]\!]$. Then we have

$$(\varphi_{K|L} \circ \widetilde{\psi_{\mathrm{Col}}})(f(Z)) = \sum_{a \in \mathcal{G}_\phi[\pi_L]} f^{\sigma_{K|L}}(a +_{\mathcal{G}_\phi} Z)$$

and

$$(\varphi_{K|L} \circ \widetilde{\mathcal{N}})(f(Z)) = \prod_{a \in \mathcal{G}_\phi[\pi_L]} f^{\sigma_{K|L}}(a +_{\mathcal{G}_\phi} Z).$$

In order to imitate the formulae from [Col79, Theorem 4, p. 98] and [Col79, Theorem 11, p. 102] we make the following definitions.

Definition 4.1.16.

$$\begin{aligned}\psi_{\mathrm{Col}} &:= \sigma_{K|L}^{-1} \circ \widetilde{\psi_{\mathrm{Col}}} = \widetilde{\psi_{\mathrm{Col}}} \circ \sigma_{K|L}^{-1},\\ \mathcal{N} &:= \sigma_{K|L}^{-1} \circ \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}} \circ \sigma_{K|L}^{-1}.\end{aligned}$$

Note that the second equality at both lines comes from Remark 4.1.14

Remark 4.1.17.
Let $f \in \mathcal{O}_K[\![Z]\!]$. Then we have

$$(\varphi_{K|L} \circ \psi_{\mathrm{Col}})(f(Z)) = \sum_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z)$$

and

$$(\varphi_{K|L} \circ \mathcal{N})(f(Z)) = \prod_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z).$$

Remark 4.1.18.

1. $\psi_{\mathrm{Col}} \circ \varphi_{K|L} = q_L$.
2. $\psi_{\mathrm{Col}}([\pi_L]_\phi \cdot f) = Z\psi_{\mathrm{Col}}(f)$ *for any* $f \in \mathcal{O}_K[\![Z]\!]$.
3. $\mathcal{N}([\pi_L]_\phi) = Z^{q_L}$.

Proof.
Because $\varphi_{K|L}$ is injective, it suffices the check the equations after applying of $\varphi_{K|L}$.

1. $$\begin{aligned}\left(\varphi_{K|L} \circ \psi_{\mathrm{Col}} \circ \varphi_{K|L}\right)(f)(Z) &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} \left(\varphi_{K|L}(f)\right)(a +_{\mathcal{G}_\phi} Z) \\ &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} (f^{\sigma_{K|L}})([\pi_L]_\phi(a +_{\mathcal{G}_\phi} Z)) \\ &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} (f^{\sigma_{K|L}})([\pi_L]_\phi(Z)) \\ &= \varphi_{K|L}(q_L f^{\sigma_{K|L}})\end{aligned}$$
2. $$\begin{aligned}\left(\varphi_{K|L} \circ \psi_{\mathrm{Col}}\right)([\pi_L](Z) \cdot f(Z)) &= \sum_{a \in \mathcal{G}_\phi[\pi_L]} [\pi_L]_\phi(a +_{\mathcal{G}_\phi} Z) f(a +_{\mathcal{G}_\phi} Z) \\ &= [\pi_L]_\phi(Z) \cdot \sum_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z) \\ &= \varphi_{K|L}(Z) \cdot \left((\varphi_{K|L} \circ \psi_{\mathrm{Col}})(f(Z))\right) \\ &= \varphi_{K|L}(Z \cdot \psi_{\mathrm{Col}}(f(Z))).\end{aligned}$$
3. $$\begin{aligned}\left(\varphi_{K|L} \circ \mathcal{N}\right)([\pi_L]_\phi(Z)) &= \prod_{a \in \mathcal{G}_\phi[\pi_L]} [\pi_L]_\phi(a +_{\mathcal{G}_\phi} Z) \\ &= \prod_{a \in \mathcal{G}_\phi[\pi_L]} [\pi_L]_\phi(Z) \\ &= \prod_{a \in \mathcal{G}_\phi[\pi_L]} \varphi_{K|L}(Z) \\ &= \varphi_{K|L}(Z^{q_L}).\end{aligned}$$

□

Remark 4.1.19.
Recall that $[\pi_L]_\phi(Z) \in Z\mathcal{O}_L[\![Z]\!]$. *Therefore, for any* $f \in \mathcal{O}_K[\![Z]\!][Z^{-1}]$ *we can find an* $n(f) \in \mathbb{N}_0$, *such that* $[\pi_L]_\phi^{n(f)} \cdot f \in \mathcal{O}_K[\![Z]\!]$. *Together with* Remark 4.1.18 *this allows us to extend* ψ_{Col} *to an* $\mathcal{O}_L$*-linear endomorphism*

$$\begin{aligned}\psi_{\mathrm{Col}} \colon \mathcal{O}_K((Z)) &\longrightarrow \mathcal{O}_K((Z)) \\ f &\longmapsto Z^{-n(f)}\psi_{\mathrm{Col}}([\pi_L]_\phi^{n(f)} f)\end{aligned}$$

as well as to extend $\mathcal{N}$ to a multiplicative map

$$\begin{aligned}\mathcal{N}\colon \mathcal{O}_K((Z)) &\longrightarrow \mathcal{O}_K((Z))\\ f &\longmapsto Z^{-q_L n(f)}\mathcal{N}([\pi_L]_\phi^{n(f)}\cdot f).\end{aligned}$$

Proof.
Let $f \in \mathcal{O}_K((Z))$. We want to give an argument that the above definition is independent from the choice of $n(f)$ as long as we have $[\pi_L]_\phi^{n(f)}\cdot f \in \mathcal{O}_K[\![Z]\!]$. So, let $n, m \in \mathbb{N}_0$ with $n > m$ such that $[\pi_L]_\phi^m \cdot f \in \mathcal{O}_K[\![Z]\!]$. Then Remark 4.1.18, 2. implies

$$\begin{aligned}Z^{-n}\psi_{\mathrm{Col}}([\pi_L]_\phi^n \cdot f) &= Z^{-n}\psi_{\mathrm{Col}}([\pi_L]_\phi^{n-m}\cdot([\pi_L]_\phi^m\cdot f))\\ &= Z^{-n}Z^{n-m}\psi_{\mathrm{Col}}([\pi_L]_\phi^m\cdot f)\\ &= Z^{-m}\psi_{\mathrm{Col}}([\pi_L]_\phi^m\cdot f).\end{aligned}$$

This is the well definition of ψ_{Col}. For $\mathcal{N}$, Remark 4.1.18, 3. implies

$$\begin{aligned}Z^{-q_L n}\mathcal{N}([\pi_L]_\phi^n\cdot f) &= Z^{-q_L n}\mathcal{N}([\pi_L]_\phi^{n-m})\mathcal{N}([\pi_L]_\phi^m\cdot f)\\ &= Z^{-q_L n}Z^{q_L(n-m)}\mathcal{N}([\pi_L]_\phi^m\cdot f)\\ &= Z^{-q_L m}\mathcal{N}([\pi_L]_\phi^m\cdot f).\end{aligned}$$

□

Now fix an $\mathcal{O}_L$-generator $t_0 = (t_{0,n})_n$ of $\mathcal{T}\mathcal{G}_\phi$.

Theorem 4.1.20 (Coleman)**.**
For any norm-coherent sequence $u = (u_n)_n \in \varprojlim K_n^\times$ there exists a unique Laurent series $g_{u,t_0} \in (\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}}$ such that $\sigma_{K|L}^{-n}(g_{u,t_0}(t_{0,n})) = u_n$ for any $n \geq 1$. This defines a multiplicative isomorphism

$$\varprojlim K_n^\times \xrightarrow{\cong} (\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}},\ u \longmapsto g_{u,t_0}.$$

Proof.
See [Col79, Thm. A, p. 92; Corollary 17, p. 105–106]. Note that Coleman uses $\widetilde{\mathcal{N}}$ and therefore his condition $\widetilde{\mathcal{N}} = \sigma_{K|L}$ translates into our $\mathcal{N} = \mathrm{id}$, since $\widetilde{\mathcal{N}} = \sigma_{K|L}\circ\mathcal{N}$ by definition. □

Remark 4.1.21.

1. *The map $(\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}} \to k_K((Z))^\times$ given by reduction modulo π_L is an*

isomorphism. Hence

$$\varprojlim K_n^\times \xrightarrow{\cong} k_K((Z))^\times,\ u \longmapsto g_{u,t_0} \bmod \pi_L$$

is an isomorphism of groups.

2. *If $t_1 = c \cdot t_0$ is a second $\mathcal{O}_L$-generator of $\mathcal{T}\mathcal{G}_\phi$, then $g_{u,t_1}([c](Z)) = g_{u,t_0}(Z)$ for any $u \in \varprojlim K_n^\times$.*

Proof.

1. See [Col79, Corollary 18, p. 106]. As in Theorem 4.1.20 note, that Coleman uses $\widetilde{\mathcal{N}}$, and therefore his condition $\widetilde{\mathcal{N}} = \sigma_{K|L}$ translates into $\mathcal{N} = \mathrm{id}$.
2. Let $t_1 = c \cdot t_0$ be a second $\mathcal{O}_L$-generator of $\mathcal{T}\mathcal{G}_\phi$ and $u \in \varprojlim K_n^\times$. By definition we have $[c](t_{0,n}) = t_{1,n}$ for all $n \geq 1$. It follows

$$g_{u,t_0}(t_{0,n}) = \sigma_{K|L}^n(u_n) = g_{u,t_1}(t_{1,n}) = g_{u,t_1}([c](t_{0,n})).$$

So the uniqueness property of Theorem 4.1.20 implies $g_{u,t_0} = g_{u,t_1} \circ [c]$ as claimed.

□

Definition 4.1.22.
As in [SV15, p. 5] we introduce the "logarithmic" homomorphism

$$\begin{aligned} \Delta_{\mathrm{LT}} \colon \mathcal{O}_K[\![Z]\!]^\times &\longrightarrow \mathcal{O}_K[\![Z]\!] \\ f &\longmapsto \frac{\partial_{\mathrm{inv}}(f)}{f} = g_{\mathrm{LT}}^{-1}\frac{f'}{f}. \end{aligned}$$

Its kernel is $\mathcal{O}_K^\times$.

Remark 4.1.23.
Δ_{LT} is in fact a homomorphism. Let $f, g \in \mathcal{O}_K[\![Z]\!]^\times$. Then

$$\Delta_{\mathrm{LT}}(f \cdot g) = g_{\mathrm{LT}}^{-1}\frac{(fg)'}{fg} = g_{\mathrm{LT}}^{-1}\frac{f'g + g'f}{fg} = g_{\mathrm{LT}}^{-1}\left(\frac{f'}{f} + \frac{g'}{g}\right) = \Delta_{\mathrm{LT}}(f) + \Delta_{\mathrm{LT}}(g).$$

Lemma 4.1.24.
On $\mathcal{O}_K[\![Z]\!]$ we have the following identities:

1. $\Delta_{\mathrm{LT}} \circ \varphi_{K|L} = \pi_L \varphi_{K|L} \circ \Delta_{\mathrm{LT}}$.
2. $\psi_{\mathrm{Col}} \circ \Delta_{\mathrm{LT}} = \pi_L \Delta_{\mathrm{LT}} \circ \mathcal{N}$.
3. $\Delta_{\mathrm{LT}}(f^{\sigma_{K|L}}) = (\Delta_{\mathrm{LT}}(f))^{\sigma_{K|L}}$ *for all $f \in \mathcal{O}_K[\![Z]\!]^\times$.*

Proof.

1. That σ is an isomorphism of $\mathcal{O}_K$ which fixes $\mathcal{O}_L$ implies $\varphi_{K|L}(g_{\mathrm{LT}}) = \varphi_L(g_{\mathrm{LT}})$ and $(f^{\sigma_{K|L}})' = (f')^\sigma$. Thereby the proof of of this statement is similar to the one of [SV15, Lemma 2.4, p. 5].

2. $$\begin{aligned}\left(\varphi_{K|L}\circ\psi_{\mathrm{Col}}\circ\Delta_{\mathrm{LT}}\right)(f) &= \sum_{a\in\mathcal{G}_\phi[\pi_L]}(\Delta_{\mathrm{LT}}(f))(a+_{\mathcal{G}_\phi}Z)\\ &= \sum_{a\in\mathcal{G}_\phi[\pi_L]}\frac{1}{g_{\mathrm{LT}}(a+_{\mathcal{G}_\phi}Z)}\frac{f'(a+_{\mathcal{G}_\phi}Z)}{f(a+_{\mathcal{G}_\phi}Z)}\\ &= \sum_{a\in\mathcal{G}_\phi[\pi_L]}\frac{1}{g_{\mathrm{LT}}(a+_{\mathcal{G}_\phi}Z)}\frac{\frac{\mathrm{d}}{\mathrm{d}Z}f(a+_{\mathcal{G}_\phi}Z)}{f(a+_{\mathcal{G}_\phi}Z)}\frac{1}{\frac{\mathrm{d}}{\mathrm{d}Z}(a+_{\mathcal{G}_\phi}Z)}\\ &= \sum_{a\in\mathcal{G}_\phi[\pi_L]}\frac{1}{\frac{\mathrm{d}}{\mathrm{d}Z}\log_{\mathrm{LT}}(a+_{\mathcal{G}_\phi}Z)}\frac{\frac{\mathrm{d}}{\mathrm{d}Z}f(a+_{\mathcal{G}_\phi}Z)}{f(a+_{\mathcal{G}_\phi}Z)}\\ &= \sum_{a\in\mathcal{G}_\phi[\pi_L]}\Delta_{\mathrm{LT}}(f(a+_{\mathcal{G}_\phi}Z))\\ &= \Delta_{\mathrm{LT}}\left(\prod_{a\in\mathcal{G}_\phi[\pi_L]}f(a+_{\mathcal{G}_\phi}Z)\right)\\ &= \Delta_{\mathrm{LT}}((\varphi_{K|L}\circ\mathcal{N})(f))\\ &= \pi_L\varphi_{K|L}((\Delta_{\mathrm{LT}}\circ\mathcal{N})(f))\\ &= \varphi_{K|L}(\pi_L(\Delta_{\mathrm{LT}}\circ\mathcal{N})(f)).\end{aligned}$$

3. As before, because $\sigma_{K|L}$ fixes $\mathcal{O}_L$, it is $g_{\mathrm{LT}}^{\sigma_{K|L}} = g_{\mathrm{LT}}$ and $(f^{\sigma_{K|L}})' = (f')^\sigma$. So it follows
$$\Delta_{\mathrm{LT}}(f^{\sigma_{K|L}}) = g_{\mathrm{LT}}^{-1}\frac{(f^{\sigma_{K|L}})'}{f^{\sigma_{K|L}}} = g_{\mathrm{LT}}^{-1}\frac{(f')^{\sigma_{K|L}}}{f^{\sigma_{K|L}}} = \left(g_{\mathrm{LT}}^{-1}\frac{f'}{f}\right)^{\sigma_{K|L}} = (\Delta_{\mathrm{LT}}(f))^{\sigma_{K|L}}.$$

$\square$

Remark 4.1.25.
Δ_{LT} restricts to a homomorphism

$$\Delta_{\mathrm{LT}}\colon\left(\mathcal{O}_K[\![Z]\!]^\times\right)^{\mathcal{N}=\mathrm{id}}\longrightarrow\mathcal{O}_K[\![Z]\!]^{\psi_{\mathrm{Col}}=\pi_L}.$$

with kernel the roots of unityμ_{q_K-1} of order dividing q_K-1.

Proof.

Let $f \in (\mathcal{O}_K[\![Z]\!]^\times)^{\mathcal{N}=\mathrm{id}}$. Then Remark 4.1.24, 2. induces

$$\begin{aligned}\psi_{\mathrm{Col}}(\Delta_{\mathrm{LT}}(f)) &= \pi_L \Delta_{\mathrm{LT}}(\mathcal{N}(f)) \\ &= \pi_L \Delta_{\mathrm{LT}}(f) \\ &= \pi_L(\Delta_{\mathrm{LT}}(f)),\end{aligned}$$

i.e we get $\psi_{\mathrm{Col}} = \pi_L$ on $\mathrm{im}(\Delta_{\mathrm{LT}})$.
Since $\mathcal{O}_K[\![Z]\!]$ has no zero divisors, it is $\Delta_{\mathrm{LT}}(f) = 0$ if and only if $f' = 0$, i.e. if and only of $f \in \mathcal{O}_K$. Therefore, the kernel of the above restriction are exactly these elements of $\mathcal{O}_K$, on which $\mathcal{N}$ is the identity. So, in the following we compute $\mathcal{O}_K^{\mathcal{N}=\mathrm{id}}$. Because of the injectivity of $\varphi_{K|L}$ it is

$$\mathcal{N}(x) = x$$

if and only if

$$\varphi_{K|L}(\mathcal{N}(x)) = \varphi_{K|L}(x)$$

for all $x \in \mathcal{O}_K^\times$ and because of Remark 4.1.17 it is $\varphi_{K|L}(\mathcal{N}(x)) = x^{q_L}$ as $\mathcal{T}\mathcal{G}_\phi[\pi_L]$ has q_L elements. Since $\varphi_{K|L}$ acts on $\mathcal{O}_K^\times$ as $\sigma_{K|L}$, it is $\mathcal{N}(x) = x$ if and only if

$$\sigma_{K|L}(x) = x^{q_L}.$$

Furthermore, since $K|L$ is unramified of degree $d_{K|L}$, the Galois automorphism $\sigma_{K|L}$ has degree $d_{K|L}$ and therefore we have $\sigma_{K|L}^{d_{K|L}} = \mathrm{id}_K$. Thus, by $d_{K|L}$-times multiplying $\sigma_{K|L}(x)$, the last equation implies

$$x = \sigma_{K|L}^{d_{K|L}}(x) = x^{\left(q_L^{d_{K|L}}\right)} = x^{q_K}.$$

In fact, since $\sigma_{K|L}$ sends a $(q_K - 1)$-st root of unity to its q_L-th power, the above equation is equivalent to $\sigma_{K|L}(x) = x^{q_L}$. So, in conclusion, we have seen, that for $x \in \mathcal{O}_K^\times$ we have $\mathcal{N}(x) = x$ if and only if x is a $q_K - 1$-st root of unity, which ends the proof. □

Clearly by definition, Δ_{LT} extends to the homomorphism

$$\begin{aligned}\Delta_{\mathrm{LT}} \colon \mathcal{O}_K((Z))^\times &\longrightarrow \mathcal{O}_K((Z)) \\ f &\longmapsto \frac{\partial_{\mathrm{inv}}(f)}{f} = g_{\mathrm{LT}}^{-1}\frac{f'}{f}.\end{aligned}$$

It's kernel is $\mathcal{O}_K^\times$ again.

Lemma 4.1.26.
The identity $\psi_{\mathrm{Col}} \circ \Delta_{\mathrm{LT}} = \pi_L \Delta_{\mathrm{LT}} \circ \mathcal{N}$ *holds true on* $\mathcal{O}_K((Z))^\times$.

Proof.
The proof is similar to the one of [SV15, Lemma 2.5, p. 6–7] by replacing φ_L with $\varphi_{K|L}$. □

Remark 4.1.27.
As before (cf. Remark 4.1.25), Δ_{LT} *restricts to a homomorphism*

$$\Delta_{\mathrm{LT}} \colon \left(\mathcal{O}_L((Z))^\times\right)^{\mathcal{N}=\mathrm{id}} \longrightarrow \mathcal{O}_L((Z))^{\psi_{\mathrm{Col}}=\pi_L}$$

with the same kernel μ_{q_K-1}.

4.2 Differential Forms and Residue Pairings

As mentioned at the beginning of Section 4.1, due to Lemma 3.4.3, it makes sense to work with the same variable over both, L and K. In order to still simplify comparisons, we will continue working with the variable Z. Therefore, let now $\mathscr{A}_L$ be the completion of $\mathcal{O}_L[\![Z]\!][Z^{-1}]$ with respect to its π_L-adic topology (in Chapter 3 we used the variable X) and let similarly $\mathscr{A}_{\mathcal{K}|L}$ be the completion of $\mathcal{O}_K[\![Z]\!][Z^{-1}]$ with respect to its π_L-adic topology. Let $\mathscr{B}_L$ and $\mathscr{B}_{\mathcal{K}|L}$ be their respective fraction fields. From the construction in Chapter 3 we then can deduce that $\mathscr{A}_{K|L}$ identifies with $\mathbf{A}_{K|L}$, as well as $\mathscr{B}_L$ with $\mathbf{B}_L$ and $\mathscr{B}_{K|L}$ with $\mathbf{B}_{K|L}$. We will also denote the Frobenii on $\mathscr{B}_L$ and $\mathscr{B}_{K|L}$ with φ_L and $\varphi_{K|L}$ respectively.

Remark 4.2.1.
Since ∂_{inv} *is continuous (cf. Remark 4.1.7), it extends to a homomorphism of* $\mathscr{B}_{K|L}$ *and for* $f \in \mathscr{B}_{K|L}$ *we still have*

$$\partial_{\mathrm{inv}}(f) = \frac{f'}{g_{\mathrm{LT}} f}.$$

Recall from Corollary 3.5.3 that $\mathscr{A}_{K|L}$ is free of degree q_L as $\varphi_{K|L}(\mathscr{A}_{K|L})$-module with basis $(1, Z, \dots, Z^{q_L-1})$ and that trace maps of totally ramified extensions are zero in the residue class field.

Definition 4.2.2.
Let Tr denote the trace map of the finite extension $\mathscr{B}_{K|L}|\varphi_{K|L}(\mathscr{B}_{K|L})$. Then define

$$\psi_{K|L} := \frac{1}{\pi_L}\varphi_{K|L}^{-1} \circ \mathrm{Tr}.$$

Remark 4.2.3.
For all $f, g \in \mathscr{B}_{K|L}$ we have

$$\psi_{K|L}(\varphi_{K|L}(f)g) = f\psi_{K|L}(g).$$

and we have

$$\psi_{K|L} \circ \varphi_{K|L} = \frac{q_L}{\pi_L}\mathrm{id}.$$

Proof.
Since $\varphi_{K|L}$ is injective it is enough to prove the assertions after applying $\varphi_{K|L}$. Let $f, g \in \mathscr{B}_{K|L}$. Then

$$\begin{aligned}
\varphi_{K|L}(\psi_{K|L}(\varphi_{K|L}(f)g)) &= \frac{1}{\pi_L}\mathrm{Tr}(\varphi_{K|L}(f)g) \\
&= \varphi_{K|L}(f)\frac{1}{\pi_L}\mathrm{Tr}(g) \\
&= \varphi_{K|L}(f)\varphi_{K|L}(\psi_{K|L}(g)) \\
&= \varphi_{K|L}(f\psi_{K|L}(g)).
\end{aligned}$$

For the other equality, we compute

$$\begin{aligned}
\varphi_{K|L}(\psi_{K|L}(\varphi_{K|L}(f))) &= \frac{1}{\pi_L}\mathrm{Tr}(\varphi_{K|L}(f)) \\
&= \frac{1}{\pi_L}\varphi_{K|L}(f)\mathrm{Tr}(1) \\
&= \frac{q_L}{\pi_L}\varphi_{K|L}(f).
\end{aligned}$$

□

Definition 4.2.4.
Let Nor denote the norm map of the extension $\mathscr{B}_{K|L}|\varphi_{K|L}(\mathscr{B}_{K|L})$. Then define

$$\mathrm{Nor}_{K|L} := \varphi_{K|L}^{-1} \circ \mathrm{Nor}.$$

Remark 4.2.5.
The restrictions of $\psi_{K|L}$ and $\mathcal{N}_{K|L}$ to $\mathscr{B}_L$ are denoted by ψ_L and $\mathcal{N}_L$ respectively.

These then are exactly the maps from [SV15, p. 8].

We then have the same remark as in [SV15, Remark 3.2, p. 8–9]. In the proof one just has to do the following adaptions:
Replace φ_L, ψ_L, ψ_{Col} and $\mathcal{N}_L$ by $\varphi_{K|L}$, ψ_L, our definition of ψ_{Col} and $\mathcal{N}_{K|L}$ respectively. Sometimes there is also a $\sigma_{K|L}$ involved.

Remark 4.2.6.

1. $\psi_{K|L}(\mathscr{A}_{K|L}) \subseteq \mathscr{A}_{K|L}$ *and* $\mathcal{N}_{K|L}(\mathscr{A}_{K|L}) \subseteq \mathscr{A}_{K|L}$.
2. *On* $\mathcal{O}_K[\![Z]\!]$ *we have* $\psi_{K|L} = \pi_L^{-1}\psi_{\mathrm{Col}}$ *and* $\mathcal{N}_{K|L} = \mathcal{N}$.
3. *On* $\mathscr{B}_{K|L}$ *we have* $\varphi_{K|L} \circ \psi_{K|L} \circ \partial_{\mathrm{inv}} = \partial_{\mathrm{inv}} \circ \varphi_{K|L} \circ \psi_{K|L}$.
4. $\mathcal{N}_{K|L}(f)([c]_\phi Z) = \mathcal{N}_{K|L}(f([c]_\phi Z))$ *for any* $c \in \mathcal{O}_L^\times$ *and* $f \in \mathscr{B}_{K|L}$.
5. $\mathcal{N}_{K|L}(f) \equiv f \bmod \pi_L \mathscr{A}_{K|L}$ *for any* $f \in \mathscr{A}_{K|L}$.
6. *If* $f \in \mathscr{A}_{K|L}$ *satisfies* $f \equiv 1 \bmod \pi_L^m \mathscr{A}_{K|L}$ *for some* $m \geq 1$ *then* $\mathcal{N}_{K|L}(f) \equiv 1 \bmod \pi_L^{m+1}\mathscr{A}_{K|L}$.
7. $(\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}} = (\mathscr{A}_{K|L}^\times)^{\mathcal{N}_{K|L}=\mathrm{id}}$.

Corollary 4.2.7.
With the above Remark 4.2.6, 7. *the isomorphism of* Theorem 4.1.20 *becomes*

$$\varprojlim K_n^\times \cong (\mathscr{A}_{K|L}^\times)^{\mathcal{N}_{K|L}=\mathrm{id}}.$$

Definition 4.2.8.
Let $\Omega^1_{\mathscr{A}_{K|L}} := \mathscr{A}_{K|L}\mathrm{d}Z$ be the **differential forms** , which are free and of rank one over $\mathscr{A}_{K|L}$. Let furthermore

$$\mathrm{Res}\colon \Omega^1_{\mathscr{A}_{K|L}} \longrightarrow \mathcal{O}_K, \ \left(\sum_i a_i Z^i\right)\mathrm{d}Z \longmapsto a_{-1}$$

be the **residue homomorphism**.

Remark 4.2.9.
The homomorphism Res *is continuous for the weak topology on* $\Omega^1_{\mathscr{A}_{K|L}}$.

Proof.
The preimage of $\pi_L^m\mathcal{O}_K$ contains $\mathcal{O}_K[\![Z]\!]$ and $\pi_L^m\mathscr{A}_{K|L}$ and therefore in particular $X^m\mathcal{O}_K[\![Z]\!] + \pi_L^m\mathscr{A}_{K|L}$. Note that $\mathbf{A}_{K|L}^+$ corresponds to $\mathcal{O}_K[\![Z]\!]$. □

Remark 4.2.10.
The homomorphism Res *does not depend on the choice of the variable.*

Proof.
This is [SV15, Remark 3.4, p. 10–11]. □

The following Remark explains how the Frobenius from $K|L$ interacts with the residue homomorphism. It's not spectacular, but it leads later on to some changes in the equations we deduce in the same way as [SV15, p. 16–18].

Remark 4.2.11.
For $f \in \mathscr{A}_{K|L}$ *we have*

$$\mathrm{Res}(f^{\sigma_{K|L}}) = \sigma_{K|L}(\mathrm{Res}(f)).$$

Definition 4.2.12.
We define the **residue pairing** by

$$\mathscr{A}_{K|L} \times \Omega^1_{\mathscr{A}_{K|L}} \longrightarrow \mathcal{O}_K, \ (f, \omega) \longmapsto \mathrm{Res}(f\omega).$$

Remark 4.2.13.
The residue pairing is jointly continuous.

Remark 4.2.14.
The above residue pairing from Definition 4.2.12 *induces for every* $m \geq 1$ *a pairing*

$$\mathscr{A}_{K|L}/\pi_L^m \mathscr{A}_{K|L} \times \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^m \Omega^1_{\mathscr{A}_{K|L}} \longrightarrow K/\mathcal{O}_K, \ (f, \omega) \longmapsto \pi_L^{-m}\mathrm{Res}(f\omega) \bmod \mathcal{O}_K.$$

This again is continuous.

Definition 4.2.15.
In the following, we will denote by $\mathrm{Hom}^{\mathrm{cts}}$ the set of **continuous homomorphisms** between two objects.

Remark 4.2.16.
As in [SV15, (14), p. 11] *the above* Remark 4.2.14 *together with* [Bou07, X.28, Theorem 3] *induces a continuous homomorphism of* $\mathcal{O}_K$*-modules*

$$\begin{aligned} \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^m \Omega^1_{\mathscr{A}_{K|L}} &\longrightarrow \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_K}(\mathscr{A}_{K|L}/\pi_L^m \mathscr{A}_{K|L}, K/\mathcal{O}_K), \\ \omega &\longmapsto [f \mapsto \pi_L^{-m}\mathrm{Res}(f\omega) \bmod \mathcal{O}_K]. \end{aligned}$$

In particular, this is an isomorphism of topological $\mathcal{O}_K$*-modules.*

Proof.
This is similar to [SV15, Lemma 3.5, p. 11] □

Lemma 4.2.17.
Let M be a finitely generated $\mathscr{A}_{K|L}/\pi_L^m\mathscr{A}_{K|L}$-module. Then we have a topological isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}_{K|L}}(M, \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^m\Omega^1_{\mathscr{A}_{K|L}}) &\xrightarrow{\cong} \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_K}(M, K/\mathcal{O}_K) \\ F &\longmapsto \pi_L^{-m}\mathrm{Res}(F(.)) \bmod \mathcal{O}_K. \end{aligned}$$

Proof.
The proof is similar the one of [SV15, Lemma 3.6, p. 11–12]. □

Remark 4.2.18.
Since $\mathscr{A}_{K|L}$ and $\mathbf{A}_{K|L}$ are naturally isomorphic (by sending the variable Z to ω_ϕ), we have the language of $(\varphi_{K|L}, \Gamma_K)$-modules and all its results also over $\mathscr{A}_{K|L}$. We will make use of it in the following.

Definition 4.2.19.
Let $M \in \mathbf{Mod}^{\mathrm{et}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. We define the $\mathcal{O}_K$-linear endomorphism ψ_M of M by

$$\begin{aligned} \psi_M \colon M \xrightarrow{(\varphi_M^{\mathrm{lin}})^{-1}} \mathscr{A}_{K|L}\ {}_{\varphi_{K|L}}\!\otimes_{\mathscr{A}_{K|L}} M &\longrightarrow M \\ f \otimes m &\longmapsto \psi_{K|L}(f)m. \end{aligned}$$

Remark 4.2.20.
Let $M \in \mathbf{Mod}^{\mathrm{et}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. Then the endomorphism ψ_M is continuous for the weak topology and it satisfies the following formulas

$$\begin{aligned} \psi_M(\varphi_{K|L}(f)m) &= f\psi_M(m) \\ \psi_M(f(\varphi_M(m)) &= \psi_{K|L}(f)m \\ \psi_M \circ \varphi_M &= \frac{q_L}{\pi_L} \cdot \mathrm{id}_M, \end{aligned}$$

with $f \in \mathscr{A}_{K|L}$ and $m \in M$.

Proof.
That ψ_M satisfies the formulas follows immediately from the analogous formulas for $\psi_{K|L}$ and $\varphi_{K|L}$ (cf. Remark 4.2.3). The latter formula together with the fact that φ_M is open with respect to the weak topology, implies that ψ_M is continuous. □

We now skip some technical details (cf. [SV15, p. 12-.14]), which won't appear again, but they play an important role in the proof of the next Lemma. Since the proof in our case is literally the same, we also skip it here. But since this result is used later (cf. [SV15, Theorem 5.13, p. 30–31]) we wanted to list it here and say that it is still true.

Lemma 4.2.21.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ sucht that $\pi_L^m M = 0$ for some $m \in \mathbb{N}$. Then $\varphi_M - \mathrm{id}$ is a continuous and topologically strict endomorphism of M.

As in [SV15, p. 14–15] our next aim is to see that $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ has an internal Hom-functor, i.e. that for any $M, N \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ the $\mathscr{A}_{K|L}$-module $\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ is also an étale $(\varphi_{K|L}, \Gamma_K)$-module over $\mathscr{A}_{K|L}$. For this, we list the results from loc. cit., which are proved similar in our case and add some computations.

Lemma 4.2.22.
Let M, N be two finitely generated $\mathscr{A}_{K|L}$-modules. Then we have:

1. *The weak topology on $\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ coincides with the topology of pointwise convergence.*
2. *The bilinear map*
$$\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N) \times M \longrightarrow N,\ (\alpha, m) \longmapsto \alpha(m)$$
is continuous for the weak topology on all three terms.

Proof.
The proof is similar to the one of [SV15, Remark 3.13, p. 14–15] □

Proposition 4.2.23.
Let $M, N \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. Then $\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ is also an etale $(\varphi_{K|L}, \Gamma_K)$-module over $\mathscr{A}_{K|L}$ with respect to

$$\gamma(\alpha) := \gamma \circ \alpha \circ \gamma^{-1}$$
$$\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\alpha) := \varphi_N^{\mathrm{lin}} \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right) \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1}$$

for any $\gamma \in \Gamma_K$ and $\alpha \in \mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$.

Proof.
We have to prove the following claims.

1. The Γ_K-action commutes with $\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}$.
2. The Γ_K-action is continuous for the weak topology.
3. $\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ is etale.

The proofs of 2 and 3 are similar to the argumentation after [SV15, Remark 3.13, p. 15]. So 1. remains.

1. Let $\gamma \in \Gamma_K$ and $\alpha \in \mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$. Because φ_M and φ_N commute with the action of Γ_K we have
$$\varphi_M^{\mathrm{lin}} \circ \gamma = \gamma \circ \varphi_M^{\mathrm{lin}}$$
as well as the same formulas for N. So we have
$$\begin{aligned}
\gamma(\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\alpha)) &= \gamma \circ (\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\alpha)) \circ \gamma^{-1} \\
&= \gamma \circ \left(\varphi_N^{\mathrm{lin}} \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right) \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1}\right) \circ \gamma^{-1} \\
&= \left(\varphi_N^{\mathrm{lin}} \circ \gamma\right) \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right) \circ \left(\gamma^{-1} \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1}\right) \\
&= \varphi_N^{\mathrm{lin}} \circ \left(\gamma \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right) \circ \gamma^{-1}\right) \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1} \\
&= \varphi_N^{\mathrm{lin}} \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \left(\gamma \circ \alpha \circ \gamma^{-1}\right)\right) \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1} \\
&= \varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\gamma(\alpha)).
\end{aligned}$$

$\square$

Remark 4.2.24.
Let $M, N \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. Then the equality
$$\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\alpha)(\varphi_M(m)) = \varphi_N(\alpha(m))$$
holds true for all $\alpha \in \mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ and $m \in M$.

Proof.
Let $\alpha \in \mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)$ and $m \in M$. Then
$$\begin{aligned}
\varphi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,N)}(\alpha)(\varphi_M(m)) &= \left(\varphi_N^{\mathrm{lin}} \circ \left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right) \circ \left(\varphi_M^{\mathrm{lin}}\right)^{-1}\right)(\varphi_M(m)) \\
&= \varphi_N^{\mathrm{lin}}\left(\left(\mathrm{id}_{\mathscr{A}_{K|L}} \otimes \alpha\right)\left(\left(\varphi_M^{\mathrm{lin}}\right)^{-1}(\varphi_M^{\mathrm{lin}}(1 \otimes m))\right)\right) \\
&= \varphi_N^{\mathrm{lin}}((\mathrm{id} \otimes \alpha)(1 \otimes m)) \\
&= \varphi_N^{\mathrm{lin}}(1 \otimes \alpha(m)) \\
&= \varphi_N(\alpha(m)).
\end{aligned}$$

$\square$

Proposition 4.2.25.
On the $\mathscr{A}_{K|L}$-module $\Omega^1_{\mathscr{A}_{K|L}}$ is via

$$\gamma \cdot \mathrm{d}Z := [\chi_{\mathrm{LT}}(\gamma)]'(Z)\mathrm{d}Z$$
$$\varphi_{\Omega^1}(\mathrm{d}Z) := \pi_L^{-1}[\pi_L]'(Z)\mathrm{d}Z$$

a $(\varphi_{K|L}, \Gamma_K)$-module structure defined.

Proof.
First note that $[\pi_L]_\phi(Z) \equiv \pi_L Z + Z^q \bmod \pi_L$ and therefore $[\pi_L]'_\phi(Z)$ is divisible by π_L since q_L is divisible by π_L. Because Γ_K and φ_{Ω^1} operate by multiplication, these operations are continuous. The endomorphism φ_{Ω^1} is $\varphi_{K|L}$-linear by definition. We have

$$\begin{aligned}
\varphi_{\Omega^1}(\gamma(\mathrm{d}Z)) &= \varphi_{\Omega^1}([\chi_{\mathrm{LT}}(\gamma)]'(Z)\mathrm{d}Z)\\
&= \varphi_{K|L}([\chi_{\mathrm{LT}}(\gamma)]'(Z))\varphi_{\Omega^1}(\mathrm{d}Z)\\
&= \pi_L^{-1}[\chi_{\mathrm{LT}}(\gamma)]'([\pi_L](Z))[\pi_L]'(Z)\mathrm{d}Z\\
&= \pi_L^{-1}([\chi_{\mathrm{LT}}(\gamma)] \circ [\pi_L])'\mathrm{d}Z\\
&= \pi_L^{-1}([\chi_{\mathrm{LT}}(\gamma) \cdot \pi_L])'\mathrm{d}Z\\
&= \pi_L^{-1}([\pi_L \cdot \chi_{\mathrm{LT}}(\gamma)])'\mathrm{d}Z\\
&= \pi_L^{-1}([\pi_L] \circ [\chi_{\mathrm{LT}}(\gamma)])'\mathrm{d}Z\\
&= \pi_L^{-1}[\pi_L]'([\chi_{\mathrm{LT}}(\gamma)](Z))[\chi_{\mathrm{LT}}(\gamma)]'(Z)\mathrm{d}Z\\
&= \gamma(\pi_L^{-1}[\pi_L]'(Z)\mathrm{d}(Z))\\
&= \gamma(\varphi_{\Omega^1}(\mathrm{d}Z)).
\end{aligned}$$

So the operations of Γ_K and φ_{Ω^1} commute and therefore $\Omega^1_{\mathscr{A}_{K|L}}$ is a $(\varphi_{K|L}, \Gamma_K)$-module. □

Definition 4.2.26.
Let $\chi\colon \Gamma_K \to \mathcal{O}_L^\times$ be a continuous character, $W_\chi = \mathcal{O}_L t_\chi$ its representation module and $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. The **$\chi$-twisted module** of M is defined by

$$M(\chi) := M \otimes_{\mathcal{O}_L} W_\chi.$$

The $\mathscr{A}_{K|L}$-module structure is given by

$$a \cdot (m \otimes w) = (am) \otimes w.$$

Proposition 4.2.27.
Let $\chi\colon \Gamma_K \to \mathcal{O}_L^\times$ be a continuous character, $W_\chi = \mathcal{O}_L t_\chi$ its representation module, $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ and $M(\chi)$ the χ-twisted module of M. Then on $M(\chi)$ we can define operations of $\varphi_{K|L}$ and Γ_K by

$$\begin{aligned}\varphi_{M(\chi)}(m \otimes w) &\coloneqq \varphi_M(m) \otimes w,\\ \gamma \cdot (m \otimes w) &\coloneqq \chi(\gamma)((\gamma \cdot m) \otimes w).\end{aligned}$$

Then it is $M(\chi) \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ and $\psi_{M(\chi)}(m \otimes w) = \psi_M(m) \otimes w$.

Proof.
The operation of Γ_K on $M(\chi)$ is continuous because χ is continuous and the operation of Γ_K on M is continuous as well. The operation of Γ_K commutes with $\varphi_{M(\chi)}$ because the actions of Γ_K and φ_M commute and φ_M is $\varphi_{K|L}$-linear, especially is $\varphi_M(am) = a\varphi_M(m)$ for all $a \in \mathcal{O}_L$ and $m \in M$. It remains to show that the map

$$\varphi^{\text{lin}}_{M(\chi)}\colon \mathscr{A}_{K|L} \otimes_{\mathscr{A}_{K|L},\varphi_{K|L}} M(\chi) \to M(\chi),\ f \otimes m \otimes w \mapsto f \otimes \varphi_{M(\chi)}(m \otimes w)$$

is bijective. This follows immediately from the assumption that φ^{lin}_M is bijective, $M(\chi) = M \otimes W_\chi$ and

$$\begin{aligned}\varphi^{\text{lin}}_{M(\chi)}(f \otimes m \otimes w) &= f \otimes \varphi_{M(\chi)}(m \otimes w)\\ &= f \otimes \varphi_M(m) \otimes w\\ &= \varphi^{\text{lin}}_M(f \otimes m) \otimes w,\end{aligned}$$

i.e. the inverse map of $\varphi^{\text{lin}}_{M(\chi)}$ is the map

$$M(\chi) \to \mathscr{A}_{K|L} \otimes_{\mathscr{A}_{K|L},\varphi_{K|L}} M(\chi),\ m \otimes w \mapsto (\varphi^{\text{lin}}_M)^{-1}(m) \otimes w.$$

□

Remark 4.2.28.
For the character χ_{LT} we take $W_{\chi_{\text{LT}}} = T = \mathcal{O}_L t_0$ as representation module and for χ_{LT}^{-1} we take its dual, i.e. $W_{\chi_{\text{LT}}^{-1}} = T^* = \mathcal{O}_L t_0^*$ where t_0^* is the dual basis of t_0.

Proposition 4.2.29.
The map

$$\mathscr{A}_{K|L}(\chi_{\text{LT}}) \to \Omega^1_{\mathscr{A}_{K|L}},\ f \otimes t_0 \mapsto f g_{\text{LT}} \mathrm{d}Z$$

is an isomorphism of $(\varphi_{K|L}, \Gamma_K)$*-modules. Therefore* $\Omega^1_{\mathscr{A}_{K|L}}$ *is an etale* $(\varphi_{K|L}, \Gamma_K)$*-module.*

Proof.
In this proof we will call the map under consideration α. It is well defined and bijective since g_{LT} is a unit in $\mathcal{O}_L[\![Z]\!]$. So we have to show that α respects the operations of Γ_K and $\varphi_{K|L}$. Let $f \in \mathscr{A}_{K|L}$ then we have

$$\begin{aligned}
\alpha(\varphi_{\mathscr{A}_{K|L}(\chi_{\mathrm{LT}})}(f(Z) \otimes t_0)) &= \alpha(\varphi_{K|L}(f(Z)) \otimes t_0) \\
&= \alpha(f^{\sigma_{K|L}}([\pi_L](Z)) \otimes t_0) \\
&= f^{\sigma_{K|L}}([\pi_L](Z)) g_{\mathrm{LT}}(Z)\mathrm{d}Z \\
&= f^{\sigma_{K|L}}([\pi_L](Z)) \pi_L^{-1} \pi_L g_{\mathrm{LT}}(Z)\mathrm{d}Z \\
&= f^{\sigma_{K|L}}([\pi_L](Z)) g_{\mathrm{LT}}([\pi_L](Z))[\pi_L]'(Z)\pi_L^{-1}\mathrm{d}Z \\
&= \varphi_{K|L}(f(Z))\varphi_{K|L}(g_{\mathrm{LT}}(Z))\varphi_{\Omega^1}(\mathrm{d}Z) \\
&= \varphi_{\Omega^1}(f(Z)g_{\mathrm{LT}}(Z)\mathrm{d}Z) \\
&= \varphi_{\Omega^1}(\alpha(f(Z) \otimes t_0)).
\end{aligned}$$

Let additionally $\gamma \in \Gamma_K$, then

$$\begin{aligned}
\alpha(\gamma \cdot (f(Z) \otimes t_0)) &= \alpha((\chi_{\mathrm{LT}}(\gamma)\gamma \cdot f(Z)) \otimes t_0) \\
&= \chi_{\mathrm{LT}}(\gamma)(\gamma \cdot f(Z)) g_{\mathrm{LT}}(Z)\mathrm{d}Z \\
&= (\gamma \cdot f(Z)) g_{\mathrm{LT}}([\chi_{\mathrm{LT}}(\gamma)](Z))[\chi_{\mathrm{LT}}(\gamma)]'(Z)\mathrm{d}Z \\
&= (\gamma \cdot f(Z))(\gamma \cdot g_{\mathrm{LT}}(Z))(\gamma \cdot \mathrm{d}Z) \\
&= \gamma \cdot (f(Z)g_{\mathrm{LT}}(Z)\mathrm{d}Z) \\
&= \gamma \cdot \alpha(f(Z) \otimes t_0).
\end{aligned}$$

So α respects the given $\varphi_{K|L}$- and Γ_K-structures. According to Definition and Proposition 4.2.26 we have $\mathscr{A}_{K|L}(\chi_{\mathrm{LT}}) \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ and so we also have $\Omega^1_{\mathscr{A}_{K|L}} \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$. ∎

Remark 4.2.30.
The above proof showed also that $g_{\mathrm{LT}}(Z)\mathrm{d}Z$ *is* φ_{Ω^1}*-invariant, i.e. it is*

$$\varphi_{\Omega^1}(g_{\mathrm{LT}}(Z)\mathrm{d}Z) = g_{\mathrm{LT}}(Z)\mathrm{d}Z.$$

And because of $g_{\mathrm{LT}}(Z)\mathrm{d}Z = \mathrm{d}\log_{\mathrm{LT}}$ *is* $\mathrm{d}\log_{\mathrm{LT}}$ *also* φ_{Ω^1}*-invariant.*

We still follow [SV15, p. 16–18] and as it is done there, we want to deduce some rules for the computation of the residue pairing, we introduced in Definition 4.2.12.

Remark 4.2.31.
For $\hat{u} \in (\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}}$ the differential form $\frac{\mathrm{d}\hat{u}}{\hat{u}}$ is ψ_{Ω^1}-invariant. Let us first compute

$$\begin{aligned}\frac{\mathrm{d}\hat{u}}{\hat{u}} &= \frac{\hat{u}'}{\hat{u}}\mathrm{d}Z \\ &= \Delta_{\mathrm{LT}}(\hat{u})g_{\mathrm{LT}}\mathrm{d}Z \\ &= \Delta_{\mathrm{LT}}(\hat{u})\mathrm{d}\log_{\mathrm{LT}}.\end{aligned}$$

With this we get

$$\begin{aligned}\psi_{\Omega^1}\left(\frac{\mathrm{d}\hat{u}}{\hat{u}}\right) &= \psi_{\Omega^1}(\Delta_{\mathrm{LT}}(\hat{u})\mathrm{d}\log_{\mathrm{LT}}) \\ &\overset{4.2.30}{=} \psi_{\Omega^1}(\Delta_{\mathrm{LT}}(\hat{u})\varphi_{\Omega^1}(\mathrm{d}\log_{\mathrm{LT}})) \\ &= \psi_{K|L}(\Delta_{\mathrm{LT}}(\hat{u}))\mathrm{d}\log_{\mathrm{LT}} \\ &= \pi_L^{-1}\psi_{\mathrm{Col}}(\Delta_{\mathrm{LT}}(\hat{u}))\mathrm{d}\log_{\mathrm{LT}} \\ &\overset{4.1.26}{=} \Delta_{\mathrm{LT}}(\mathcal{N}(\hat{u}))\mathrm{d}\log_{\mathrm{LT}} \\ &= \Delta_{\mathrm{LT}}(\hat{u})\mathrm{d}\log_{\mathrm{LT}} \\ &= \frac{\mathrm{d}\hat{u}}{\hat{u}}\end{aligned}$$

Lemma 4.2.32.
The map $\mathrm{d}\colon \mathscr{A}_{K|L} \to \Omega^1_{\mathscr{A}_{K|L}}$ satisfies

1. $\pi_L \cdot \varphi_{\Omega^1} \circ \mathrm{d} = \mathrm{d} \circ \varphi_{K|L}$.
2. $\gamma \circ \mathrm{d} = \mathrm{d} \circ \gamma$ *for any* $\gamma \in \Gamma_K$.
3. $\pi_L \cdot \psi_{\Omega^1} \circ \mathrm{d} = \mathrm{d} \circ \psi_{K|L}$.

Proof.
This is [SV15, Lemma 3.16, p. 16]. We add some details and transfer it to our situation.

1. Let $f \in \mathscr{A}_{K|L}$. Then:

$$\begin{aligned}\varphi_{\Omega^1}(\mathrm{d}f) &= \varphi_{\Omega^1}(f'\mathrm{d}Z) = \varphi_{K|L}(f')\varphi_{\Omega^1}(\mathrm{d}Z) \\ &= f'^{\sigma}([\pi_L](Z))\pi_L^{-1}[\pi_L]'(Z)\mathrm{d}Z \\ &= \pi_L^{-1}\mathrm{d}(f^{\sigma_{K|L}}([\pi_L](Z))) = \pi_L^{-1}\mathrm{d}(\varphi_{K|L}(f(Z)))\end{aligned}$$

2. Let $f \in \mathscr{A}_{K|L}$. Then:

$$\begin{aligned}\gamma \cdot \mathrm{d}f &= \gamma(f'\mathrm{d}Z) = (\gamma \cdot f')(\gamma \cdot \mathrm{d}Z)\\ &= f'([\chi_{\mathrm{LT}}(\gamma)(Z))[\chi_{\mathrm{LT}}(\gamma)]'(Z)\mathrm{d}Z\\ &= \mathrm{d}(f([\chi_{\mathrm{LT}}(\gamma)](Z))) = \mathrm{d}(\gamma \cdot f)\end{aligned}$$

3. Since φ_{Ω^1} is injective the identity in question is equivalent to

$$\varphi_{\Omega^1} \circ \psi_{\Omega^1} \circ \mathrm{d} = \mathrm{d} \circ \varphi_{K|L} \circ \psi_{K|L}$$

by the first part of this proof. From Proposition 4.2.29 we deduce

$$(\varphi_{\Omega^1} \circ \psi_{\Omega^1})(fg_{\mathrm{LT}}\mathrm{d}Z) = (\varphi_{K|L} \circ \psi_{K|L})(f)g_{\mathrm{LT}}\mathrm{d}Z$$

for all $f \in \mathscr{A}_{K|L}$ since $\mathscr{A}_{K|L}(\chi_{\mathrm{LT}})$ and $\Omega^1_{\mathscr{A}_{K|L}}$ are isomorphic as $(\varphi_{K|L}, \Gamma_K)$-modules and therefore $\mathscr{A}_{K|L}$ and $\Omega^1_{\mathscr{A}_{K|L}}$ are isomorphic as $\mathscr{A}_{K|L}$-modules equipped with an étale endomorphism $\varphi_{K|L}$ resp. φ_{Ω^1}. Let $f \in \mathscr{A}_{K|L}$. Then:

$$\begin{aligned}(\varphi_{\Omega^1} \circ \psi_{\Omega^1})(\mathrm{d}f) &= (\varphi_{\Omega^1} \circ \psi_{\Omega^1})(\partial_{\mathrm{inv}}(f)g_{\mathrm{LT}}\mathrm{d}Z)\\ &= ((\varphi_{K|L} \circ \psi_{K|L})(\partial_{\mathrm{inv}}(f)))g_{\mathrm{LT}}\mathrm{d}Z\\ &\overset{4.2.6.3}{=} \partial_{\mathrm{inv}}(\varphi_{K|L} \circ \psi_{K|L}(f))g_{\mathrm{LT}}\mathrm{d}Z\\ &\overset{(4.1.5)}{=} \mathrm{d}(\varphi_{K|L} \circ \psi_{K|L}(f)).\end{aligned}$$

□

In the following Proposition Remark 4.2.11 leads to some changes to the corresponding formulas of [SV15, Proposition 3.17, p. 16].

Proposition 4.2.33.
The residue map $\mathrm{Res}\colon \Omega^1_{\mathscr{A}_{K|L}} \to K$ *(c.f. Definition 4.2.8) satisfies the following equalities:*

1. $\mathrm{Res} \circ \varphi_{\Omega^1} = \pi_L^{-1} q_L \sigma_{K|L} \circ \mathrm{Res}$.

2. $\mathrm{Res} \circ \gamma = \mathrm{Res}$ *for all* $\gamma \in \Gamma_K$.

3. $\mathrm{Res} \circ \psi_{\Omega^1} = \sigma_{K|L}^{-1}\mathrm{Res}$.

Proof.
The proof is similar to the proof of [SV15, Proposition 3.17, p. 16]. In the reduction

step, explained before starting proving the results, one has to recall that φ_{Ω^1} acts as $\sigma_{K|L}$ on the coefficients, which leads exactly to the above formulas. $\square$

Corollary 4.2.34.
The residue pairing satisfies

$$\operatorname{Res}(f\psi_{\Omega^1}(\omega)) = \sigma_{K|L}^{-1}\operatorname{Res}(\varphi_{K|L}(f)\omega)$$

for all $f \in \mathscr{A}_{K|L}$ *and* $\omega \in \Omega^1_{\mathscr{A}_{K|L}}$.

Proof.
The proof is similar to the proof of [SV15, Corollary 3.18, p. 17]. We do it here, because we have some slightly different formulas. From the projection formula in Remark 4.2.20 we get that the left hand side is equal to

$$\operatorname{Res}(\psi_{\Omega^1}(\varphi_{K|L}(f)\omega).$$

The above Lemma 4.2.33 then says

$$\operatorname{Res}(\psi_{\Omega^1}(\varphi_{K|L}(f)\omega) = \sigma_{K|L}^{-1}\operatorname{Res}(\varphi_{K|L}(f)\omega).$$

$\square$

Proposition 4.2.35.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ *such that* $\pi_L M = 0$ *fur some* $n \geq 1$. *Then the pairing*

$$\begin{aligned} [,] = [,]_M \colon M \times \operatorname{Hom}_{\mathscr{A}_{K|L}}(M, \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}}) &\longrightarrow K/\mathcal{O}_K, \\ (m, F) &\longmapsto \pi_L^{-n}\operatorname{Res}(F(m)) \bmod \mathcal{O}_K \end{aligned}$$

satisfies the following properties.

1. *The pairing* $[,]_M$ *is jointly continuous.*

2. *The pairing* $[,]_M$ *is* Γ_K*-invariant.*

3. *Under the pairing* $[,]_M$ *the operator* ψ_M *is left adjoint to* $\varphi_{\operatorname{Hom}_{\mathscr{A}_{K|L}}(M,\Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})}$, *i.e. for all* $m \in M$ *and* $F \in \operatorname{Hom}_{\mathscr{A}_{K|L}}(M, \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})$ *there holds*

$$[\psi_M(m), F]_M = [m, \varphi_{\operatorname{Hom}_{\mathscr{A}_{K|L}}(M,\Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})}(F)]_M.$$

4. *Under the pairing* $[,]_M$ *the operator* φ_M *is left adjoint to* $\psi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,\Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})}$, *i.e. for all* $m \in M$ *and* $F \in \mathrm{Hom}_{\mathscr{A}_{K|L}}(M, \Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})$ *there holds*

$$[\varphi_M(m), F]_M = [m, \psi_{\mathrm{Hom}_{\mathscr{A}_{K|L}}(M,\Omega^1_{\mathscr{A}_{K|L}}/\pi_L^n\Omega^1_{\mathscr{A}_{K|L}})}(F)]_M.$$

Proof.
The proof is similar to the argumentation between [SV15, Corollary 3.18, p. 17] and [SV15, Proposition 3.19, p. 17] and the proof of [SV15, Proposition 3.19, p. 17–18]. Note that their (17) is proven here (c.f. Remark 4.2.24). □

Remark 4.2.36.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathscr{A}_{K|L})$ *such that* $\pi_L^n M = 0$ *fur some* $n \geq 1$*. Then the pairing*

$$\begin{aligned}[,\rangle = [,\rangle_M \colon M \times \mathrm{Hom}_{\mathscr{A}_{K|L}}(M, \mathscr{A}_{K|L}(\chi_{\mathrm{LT}})/\pi_L^n\mathscr{A}_{K|L}(\chi_{\mathrm{LT}})) &\longrightarrow K/\mathcal{O}_K,\\ (m,F) &\longmapsto \pi_L^{-n}\mathrm{Res}(F(m)g_{\mathrm{LT}}\mathrm{d}Z) \bmod \mathcal{O}_K\end{aligned}$$

satisfies analogous properties to the ones of Proposition 4.2.35.

Proof.
This follows with Proposition 4.2.29 from Proposition 4.2.35. □

Since we mentioned in Remark 4.2.18 that the language of $(\varphi_{K|L}, \Gamma_K)$-modules translates from $\mathbf{A}_{K|L}$ to $\mathscr{A}_{K|L}$ and we explained the theory in detail over $\mathbf{A}_{K|L}$, we will skip [SV15, Section 4, p. 18–23] in our discussion, since there is nothing new to discover. The only thing we want to mention is that the above Remark 4.2.36 translates into to language of $\mathbf{A}_{K|L}$. For further applications, we will state it here.

Remark 4.2.37.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ *such that* $\pi_L^n M = 0$ *fur some* $n \geq 1$*. Then the pairing*

$$\begin{aligned}[,\rangle = [,\rangle_M \colon M \times \mathrm{Hom}_{\mathbf{A}_{K|L}}(M, \mathbf{A}_{K|L}(\chi_{\mathrm{LT}})/\pi_L^n\mathbf{A}_{K|L}(\chi_{\mathrm{LT}})) &\longrightarrow K/\mathcal{O}_K,\\ (m,F) &\longmapsto \pi_L^{-n}\mathrm{Res}(F(m)g_{\mathrm{LT}}\mathrm{d}Z) \bmod \mathcal{O}_K\end{aligned}$$

satisfies analogous properties to the ones of Proposition 4.2.35.

4.3 Local Tate Duality and Iwasawa Cohomology

This section is nearly exactly [SV15, Section 5, p. 23–31]. Just for completeness we want to list the results, which we will need later on. Note that [SV15, Remark 5.1, p. 23] was also proven here (cf. Lemma 5.1.1).

Definition 4.3.1.
Let M be a topological $\mathcal{O}_L$-module. The **Pontrjagin dual** of M is defined as

$$M^\vee := \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(M, L/\mathcal{O}_L).$$

It is always equipped with the compact-open topology.
Note that as in [SV15, Lemma 5.3, p. 24–25] we can prove for topological $\mathcal{O}_K$-modules:

$$M^\vee \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_K}(M, K/\mathcal{O}_K).$$

Proposition 4.3.2 (Pontrjagin duality)**.**
The functor $-^\vee$ defines an involuntary contravariant autoequivalence of the category of (Hausdorff) locally compact linear-topological $\mathcal{O}_L$-modules.
In particular, for such a module M there is a canonical isomorphism

$$M \xrightarrow{\cong} (M^\vee)^\vee.$$

Proof.
This is [SV15, Proposition 5.4, p. 25–26]. □

Remark 4.3.3.
Let $M_0 \overset{\alpha}{\to} M \overset{\beta}{\to} M_1$ be a sequence of locally compact linear- topological $\mathcal{O}_K$-modules such that $\mathrm{im}(\alpha) = \ker(\beta)$ and β is topologically strict with closed image. Than the dual sequence

$$M_1^\vee \overset{\beta^\vee}{\to} M^\vee \overset{\alpha^\vee}{\to} M_0^\vee$$

is exact as well.

Proof.
The proof is similar to the one of [SV15, Remark5.5, p. 27]. □

Remark 4.3.4.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ of finite length and $n \geq 1$ sucht that $\pi_L^n V = 0$. Then there is a natural isomorphism of topological groups:

$$\mathcal{M}_{K|L}(V)^\vee \cong \mathcal{M}_{K|L}(V^\vee(\chi_{\mathrm{LT}})).$$

This isomorphism identifies $\psi_{\mathcal{M}_{K|L}(V^\vee(\chi_{\mathrm{LT}}))}$ with $\varphi^\vee_{\mathcal{M}_{K|L}(V)}$.

Proof.
This is [SV15, Remark 5.6, p. 27] □

Proposition 4.3.5 (Local Tate duality)**.**
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$, $n \geq 1$ such that $\pi_L^n V = 0$ and E a finite extension of K. Then the cup product and the local invariant map induce perfect pairings of finite $\mathcal{O}_L$-modules

$$H^i(G_E, V) \times H^{2-i}(G_E, \mathrm{Hom}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p(1))) \to H^2(G_E, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathbb{Q}_p/\mathbb{Z}_p$$

and

$$H^i(G_E, V) \times H^{2-i}(G_E, \mathrm{Hom}_{\mathcal{O}_L}(V, L/\mathcal{O}_L(1))) \to H^2(G_E, L/\mathcal{O}_L(1)) = L/\mathcal{O}_L.$$

There $-(1)$ denotes the twist by the cyclotomic character.
This means that there are conical isomorphisms

$$H^i(G_E, V) \cong H^{2-i}(K, V^\vee(1))^\vee.$$

Proof.
This is [SV15, Proposition 5.7, p. 27–28], where [Ser73, Theorem 2, p. 91–92] is applied. Because the latter is slightly different formulated, we want to check it's compatibility here:
Serre defines the dual as $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_p}(V, \mu)$, where μ is the union of all roots of unity. The condition $\pi_L^n V = 0$ implies that V is also killed by a power of p. This means, that the image of each homomorphism $V \to \mu$ is contained in the set μ_p of p-power roots of unity. As Abelian Group μ_p is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and as G_K-module to the Tate twist $\mathbb{Q}_p/\mathbb{Z}_p(1)$. Since G_K acts trivially on L we obtain together with [SV15, Lemma 5.3, p. 24–25]

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_p}(V, \mu) \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(V, L/\mathcal{O}_L(1)).$$

Therefore the first pairing is [Ser73, Theorem 2, p. 91–92]. □

Definition 4.3.6.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. The **generalized Iwasawa cohomology of** V is defined by

$$H^i_{\mathrm{Iw}}(K_\infty|K, V) := \varprojlim_{K \subseteq E \subseteq K_\infty} H^i(G_E, V).$$

We always consider these modules as Γ_K-modules.

Remark 4.3.7.
Let $E|K$ a finite extension contained in K_∞. Then there is an isomorphism of

$\mathcal{O}_L$-modules:

$$\varprojlim_{E\subseteq E'\subseteq K_\infty} H^i(G_{E'},V)\cong H^i_{\mathrm{Iw}}(K_\infty|K,V).$$

Proof.
The claim follows immediately from the fact, that the set $\{E'|E \text{ finite} \mid E'\subseteq K_\infty\}$ is cofinal in the set $\{E'|K \text{ finite} \mid E'\subseteq K_\infty\}$. □

Lemma 4.3.8.
Let $V\in\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. Then we have

$$H^i_{\mathrm{Iw}}(K_\infty|K,V)\cong H^i(G_K,\mathcal{O}_L[\![\Gamma_K]\!]\otimes_{\mathcal{O}_L}V).$$

Proof.
The proof is similar to the one of [SV15, Lemma 5.8, p. 28–29]. □

Lemma 4.3.9.
$V\mapsto H_{\mathrm{Iw}}(K_\infty|K,V)$ defines a δ-functor on $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$.

Proof.
Replace Γ_L by Γ_K in the proof of [SV15, Lemma 5.9, p. 29]. □

Remark 4.3.10.
Let $V,V_0\in\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ such that V_0 is $\mathcal{O}_L$-free and G_K acts through its factor Γ_K on V_0. Then there is a natural isomorphism

$$H^i_{\mathrm{Iw}}(K_\infty|K,V\otimes_{\mathcal{O}_L}V_0)\cong H^i_{\mathrm{Iw}}(K_\infty|K,V)\otimes_{\mathcal{O}_L}V_0.$$

Remark 4.3.11.
Let $V\in\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ of finite length. Then there is an isomorphism

$$H^i_{\mathrm{Iw}}(K_\infty|K,V)\cong H^i(H_K,V^\vee(1))^\vee.$$

Note that $H_K=G_{K_\infty}$.

Proof.
From Proposition 4.3.5 we deduce

$$H^i(G_{K_n},V)\cong H^{2-i}(G_{K_n},V^\vee(1))^\vee$$

for every $n \in \mathbb{N}$. Taking projective limits gives us

$$\begin{aligned}
H^i_{\mathrm{Iw}}(K_\infty|K, V) &= \varprojlim H^i(G_{K_n}, V) \\
&= \varprojlim H^{2-i}(G_{K_n}, V^\vee(1))^\vee \\
&= \varprojlim \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(H^{2-i}(G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\varinjlim H^{2-i}(G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(H^{2-i}(\varprojlim G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(H^{2-i}(H_K, V^\vee(1)), L/\mathcal{O}_L) \\
&= H^{2-i}(H_K, V^\vee(1))^\vee.
\end{aligned}$$

□

Lemma 4.3.12.

1. $H^i_{\mathrm{Iw}}(K_\infty|K, V) = 0$ *for* $i \neq 1, 2$.
2. $H^2_{\mathrm{Iw}}(K_\infty|K, V)$ *is finitely generated as* $\mathcal{O}_L$*-module.*
3. $H^1_{\mathrm{Iw}}(K_\infty|K, V)$ *is finitely generated as* $\mathcal{O}_L[\![\Gamma_K]\!]$*-module.*

Proof.
The proof is similar to the one of [SV15, Lemma 5.12, p. 29–30]. □

Theorem 4.3.13.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$, $\tau = \chi_{\mathrm{cyc}}\chi_{\mathrm{LT}}^{-1}$ *and* $\psi = \psi_{\mathcal{M}_{K|L}(V(\tau^{-1}))}$. *Then we have an exact sequence*

$$0 \longrightarrow H^1_{\mathrm{Iw}}(K_\infty|K, V) \longrightarrow \mathcal{M}_{K|L}(V(\tau^{-1})) \xrightarrow{\psi-1} \mathcal{M}_{K|L}(V(\tau^{-1})) \longrightarrow H^2_{\mathrm{Iw}}(K_\infty|K, V) \longrightarrow 0,$$

which is functorial in V.
Furthermore, each occurring map is continuous and $\mathcal{O}_L[\![\Gamma_K]\!]$*-equivariant.*

Proof.
The proof for the exactness of the sequence is similar to the one of [SV15, Theorem 5.13, p. 30–31]. The proof for the continuity and the $\mathcal{O}_L[\![\Gamma_K]\!]$-equivariance is similar to the one of [SV15, Remark 5.14, p. 31]. □

4.4 The Kummer Map

The next topic in [SV15, Section 6, p. 31–34] is the formulation of a reciprocity law and then to proof it in the following section. We imitate the ideas and constructions from loc. cit., and explain where the changes are, in order to transform the proof to our situation. First, we look at the Kummer isomorphism

$$\kappa\colon A(K_\infty) \coloneqq \varprojlim_{n,m} K_n^\times / K_n^{\times p^m} \cong H^1_{\mathrm{Iw}}(K_\infty|K, \mathbb{Z}_p(1))$$

which then leads to the twisted Kummer isomorphism

$$A(K_\infty) \otimes_{\mathbb{Z}_p} T^* \xrightarrow[\cong]{\kappa \otimes \mathrm{id}_{T^*}} H^1_{\mathrm{Iw}}(K_\infty|K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} T^* \xrightarrow[\cong]{4.3.10} H^1_{\mathrm{Iw}}(K_\infty|K, \mathcal{O}_L(\tau)).$$

[SV15, Theorem 5.13, p. 30–31] (resp. Theorem 4.3.13) then gives us the isomorphism

$$\mathrm{Exp}^*\colon H^1_{\mathrm{Iw}}(K_\infty|K, \mathcal{O}_L(\tau)) \cong \mathcal{M}_{K|L}(\mathcal{O}_L)^{\psi=1} = \mathbf{A}_{K|L}^{\psi_{K|L}=1}.$$

Then, combining the twisted Kummer isomorphism with Exp^* gives us the homomorphism

$$\begin{aligned} \nabla\colon (\varprojlim_n K_n^\times) \otimes_{\mathbb{Z}_p} T^* &\longrightarrow \mathbf{A}_{K|L}^{\psi=1} \\ u \otimes a t_0^* &\longmapsto a \frac{\partial_{\mathrm{inv}}(g_{u,t_0})(\iota_{\mathrm{LT}}(t_0))}{g_{u,t_0}(\iota_{\mathrm{LT}}(t_0))}. \end{aligned}$$

This homomorphism ∇ is well defined:
Theorem 4.1.20 says, that for $u \in \varprojlim K_n^\times$ the power series $g_{u,t_0} \in (\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}}$ is unique. Since $\Delta_{\mathrm{LT}}(f) = \partial_{\mathrm{inv}}(f)/f$ by definition we deduce from Remark 4.1.27, that $\Delta_{\mathrm{LT}}(g_{u,t_0}) \in \mathcal{O}_L((Z))^{\psi_{\mathrm{Col}}=\pi_L}$. Remark 4.2.6 says that $\psi_{K|L} = \pi_L^{-1}\psi_{\mathrm{Col}}$, therefore we have $\psi_{K|L}(\Delta_{\mathrm{LT}}(g_{u,t_0})) = \Delta_{\mathrm{LT}}(g_{u,t_0})$, i.e. the image of ∇ is contained in $\mathscr{A}_{K|L}^{\psi_{K|L}=1}$.

Remark 4.4.1.
The homomorphism ∇ is independent of the choice of t_0.

Proof.
It's the same proof as in [SV15, Remark 6.1, p. 32]. □

Theorem 4.4.2.
The following diagram is commutative:

$$\begin{tikzcd}
(\varprojlim_n K_n^\times) \otimes_{\mathbb{Z}_p} T^* \arrow[rr, "-\kappa\otimes \mathrm{id}_{T^*}", "\cong"'] \arrow[dr, "\nabla"'] & & H^1_{\mathrm{Iw}}(K_\infty|K, \mathcal{O}_L(\tau)) \arrow[dl, "\cong", "\mathrm{Exp}^*"'] \\
& \mathbf{A}_{K|L}^{\psi=1}. &
\end{tikzcd}$$

By rec: $(\varprojlim K_n^\times) \to H_K^{\mathrm{ab}}(p)$ we denote the map into the maximal abelian pro-p quotient $H_K^{\mathrm{ab}}(p)$ of H_K induced by the reciprocity homomorphisms of local class field theory for the intermediate extensions K_n. By $\mathrm{rec}_{\mathbf{E}_K}: \mathbf{E}_{K|L}^\times \to H_K^{\mathrm{ab}}(p)$ we denote the reciprocity homomorphism in characteristic p.
As explained in [SV15, p. 33] the proof of Theorem 4.4.2 then reduces to the following case.

Proposition 4.4.3.
For any $z \in \mathbf{A}_{K|L}$ and $u \in \mathbf{E}_{K|L}^\times$ with unique lift $\hat{u} \in (\mathbf{A}_{K|L}^\times)^{\mathcal{N}=\mathrm{id}}$ we have

$$\mathrm{Res}\left(z\frac{\mathrm{d}\hat{u}}{\hat{u}}\right) = \partial_\varphi(z)(\mathrm{rec}_{\mathbf{E}_{K|L}}(u)),$$

where ∂_φ is the connecting homomorphism

Since the connecting homomorphism for $V = \mathcal{O}_L$ induces, by reduction modulo $\pi_L^n\mathcal{O}_L$, the corresponding connecting homomorphism for $V = \mathcal{O}_L/\pi_L^n\mathcal{O}_L$, it suffices to prove the identity in Proposition 4.4.3 modulo $\pi_L^n\mathcal{O}_L$ for any $n \geq 1$. Furthermore, for every $\hat{u} \in (\mathscr{A}_{K|L}^\times)^{\mathcal{N}=\mathrm{id}}$ the differential form $\frac{\mathrm{d}\hat{u}}{\hat{u}}$ is ψ_{Ω^1}-invariant (c.f. Remark 4.2.31) and by the adjointness of ψ_{Ω^1} and $\varphi_{K|L}$ (c.f. Remark 4.2.34) we obtain

$$\mathrm{Res}\left(\varphi_{K|L}^m(z)\frac{\mathrm{d}\hat{u}}{\hat{u}}\right) = \sigma_{K|L}^m \mathrm{Res}\left(z\psi_{\Omega^1}\left(\frac{\mathrm{d}\hat{u}}{\hat{u}}\right)\right) = \sigma_{K|L}^m \mathrm{Res}\left(z\frac{\mathrm{d}\hat{u}}{\hat{u}}\right)$$

for any $m \geq 1$. Therefore, in order to prove Proposition 4.4.3 and Theorem 4.4.2 it suffices to prove the following Lemma. After the Lemma, we explain how the proof of [SV15, Lemma 7.18, p. 43–44], which is the analogous statement for $\mathbf{A}_L$, transforms to our situation.

Lemma 4.4.4.
For any $z \in \mathbf{A}_{K|L}$ and $u \in \mathbf{E}_{K|L}^\times$ with unique lift $\hat{u} \in (\mathbf{A}_{K|L}^\times)^{\mathcal{N}=\mathrm{id}}$ we have

$$\mathrm{Res}\left(\varphi_{K|L}^{n-1}(z)\frac{\mathrm{d}\hat{u}}{\hat{u}}\right) \equiv \sigma_{K|L}^{n-1}(\partial_\varphi(\mathrm{rec}_{\mathbf{E}_{K|L}}(u))) \bmod \pi_L^n\mathcal{O}_K,$$

for all $n \geq 1$.

So, in comparison to [SV15, Section 7, p. 34–44] wo have to explain where to find the Frobenius on the right hand side. Taking a closer look at loc. cit., on sees that all the hard work was done there, we just have to identify it. In particular, [SV15, Remark 7.3, p. 35–36], [SV15, Lemma 7.4, p. 36], [SV15, Lemma 7.5, p. 36] and the discussion at [SV15, p. 37] are exactly the same for $\mathbf{A}_{K|L}$ instead of $\mathbf{A}_L$. For a better clarity of the presentation, we summarize the results from loc. cit. in the following Proposition.

Proposition 4.4.5.
For every $n \geq 1$*, there exist unique* $\mathcal{O}_K$*-linear homomorphisms*

$$\begin{aligned} \alpha_n\colon \mathbf{A}_{K|L} &\longrightarrow W_n(\mathbf{E}_{K|L})_L, \\ \overline{\alpha}_n\colon \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L} &\longrightarrow W_n(\mathbf{E}_{K|L})_L. \end{aligned}$$

such that $\overline{\alpha}_n$ *is injective and the following diagram commutes*

$$\begin{tikzcd} W_n(\mathbf{A}_{K|L})_L \arrow[r, "\Phi_{n-1}"] \arrow[d, "W_n(\mathrm{pr})_L"'] & \mathbf{A}_{K|L} \arrow[d, "\alpha_n"] \\ W_n(\mathbf{E}_{K|L})_L \arrow[r, "\mathrm{Fr}^{n-1}"] & W_n(\mathbf{E}_{K|L})_L. \end{tikzcd}$$

Furthermore, for every $n \geq 1$*, it exists a unique* $\mathcal{O}_K$*-linear homomorphism*

$$w_{n-1}\colon W_n(\mathbf{E}_{K|L})_L \longrightarrow \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}$$

such that the following diagram commutes

$$\begin{tikzcd} W_n(\mathbf{A}_{K|L})_L \arrow[r, "\Phi_{n-1}"] \arrow[d, "W_n(\mathrm{pr})_L"'] & \mathbf{A}_{K|L} \arrow[d, "\mathrm{pr}"] \\ W_n(\mathbf{E}_{K|L})_L \arrow[r, "w_{n-1}"] & \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L} \end{tikzcd}$$

and the following equalities hold

$$\begin{aligned} \overline{\alpha}_n \circ w_{n-1} &= \mathrm{Fr}^{n-1}, \\ w_{n-1} \circ \overline{\alpha}_n &= \varphi_{K|L}^{n-1}, \\ w_{n-1} \circ \alpha_n &= \mathrm{pr} \circ \varphi_{K|L}^{n-1}. \end{aligned}$$

Here the first equality is an equality of endomorphisms of $W_n(\mathbf{E}_{K|L})_L$*, the second is* *one of endomorphisms of* $\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}$ *and the last one is a homomorphism from* $\mathbf{A}_{K|L}$ *to* $\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}$.

The discussion on the following pages (to be precise: [SV15, p. 37–43]) is the same for $\mathbf{A}_{K|L}$ instead of $\mathbf{A}_L$. The change then comes in the last equality of the last line in the proof of [SV15, Lemma 7.18]. Roughly, one uses there that $w_{n-1} \circ \overline{\alpha}_n = \varphi_L^{n-1}$ on $\mathbf{A}_L/\pi_L^n\mathbf{A}_L$ and therefore it is the identity for elements coming from $\mathcal{O}_L$, what is the case there. For an element $y \in \mathcal{O}_K$ we get with the last equality in the above Proposition 4.4.5

$$(w_{n-1} \circ \alpha_n)(y) = \varphi_{K|L}^{n-1}(y) \bmod \pi_L^n = \sigma_{K|L}^{n-1}(y) \bmod \pi_L^n.$$

This is exactly the desired power of $\sigma_{K|L}$ from Lemma 4.4.4, which did not occur in [SV15, Lemma 7.18, p. 43–44], since the Frobenius is equal to the identity on the base field.

Chapter 5

Galois cohomology in terms of Lubin-Tate (φ, Γ)-modules

We keep the notation from Chapter 3. Recall from Theorem 3.2 (resp. from [Sch17, p. 113-114]) that $\mathbf{E}_L^{\mathrm{sep}}$ is the residue class field of $\mathbf{A}$ and $\mathbf{E}_L^{\mathrm{sep},+}$ is the residue class field of $\mathbf{A}^+$.

5.1 Description with φ

The goal of this section is to compute Galois cohomology from the generalized φ-Herr complex, which is related to $\varphi_{K|L}$ and Γ_K.

Lemma 5.1.1.

1. *The following sequences are exact:*

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}^+ \longrightarrow 0.$$

2. *Let $E \mid L$ be a finite extension. For every $n \in \mathbb{N}$ the maps*

$$\varphi_{E|L} - \mathrm{id}\colon \omega_\phi^n \mathbf{E}_E^+ \longrightarrow \omega_\phi^n \mathbf{E}_E^+,$$

$$\mathrm{Fr} - \mathrm{id}\colon \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \longrightarrow \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$$

are isomorphisms.

3. *For every $n \in \mathbb{N}$ the map*

$$\mathrm{Fr} - \mathrm{id}\colon \omega_\phi^n \mathbf{A}^+ \longrightarrow \omega_\phi^n \mathbf{A}^+$$

is an isomorphism.

Proof.

1. We start with the sequence

$$0 \longrightarrow k_L \longrightarrow \mathbf{E}_L^{\mathrm{sep}} \xrightarrow{x \mapsto x^{q_L} - x} \mathbf{E}_L^{\mathrm{sep}} \longrightarrow 0,$$

and claim that it is exact. Recall that $\mathrm{Fr}(x) \equiv x^{q_L} \bmod \pi_L$ holds for all $x \in \mathbf{A}$ by definition. The inclusion $\mathcal{O}_L \hookrightarrow \mathbf{A}$ induces the inclusion $k_L \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}$ and we have

$$\ker(\mathrm{Fr} - \mathrm{id}) = \{x \in \mathbf{E}_L^{\mathrm{sep}} \mid x^{q_L} - x\} = k_L.$$

It remains to check that $\mathrm{Fr} - \mathrm{id}$ is surjective on $\mathbf{E}_L^{\mathrm{sep}}$. But since the polynomial $X^{q_L} - X - \alpha$ is separable for every $\alpha \in \mathbf{E}_L^{\mathrm{sep}}$ and $\mathbf{E}_L^{\mathrm{sep}}$ is separably closed by definition this follows immediately.

Now suppose that the sequence

$$0 \longrightarrow \mathcal{O}_L/\pi_L^n \mathcal{O}_L \longrightarrow \mathbf{A}/\pi_L^n \mathbf{A} \xrightarrow{\varphi_L - \mathrm{id}} \mathbf{A}/\pi_L^n \mathbf{A} \longrightarrow 0$$

is exact for $n \geq 1$ and consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_L/\pi_L^n \mathcal{O}_L \arrow[r] & \mathbf{A}/\pi_L^n \mathbf{A} \arrow[r, "\varphi_L - \mathrm{id}"] & \mathbf{A}/\pi_L^n \mathbf{A} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_L/\pi_L^{n+1} \mathcal{O}_L \arrow[r] \arrow[u, two heads] & \mathbf{A}/\pi_L^{n+1} \mathbf{A} \arrow[r, "\varphi_L - \mathrm{id}"] \arrow[u, two heads] & \mathbf{A}/\pi_L^{n+1} \mathbf{A} \arrow[r] \arrow[u, two heads] & 0.
\end{tikzcd}$$

Our aim is to show that the second sequence is exact. The kernel of the homomorphism $\mathcal{O}_L \hookrightarrow \mathbf{A} \twoheadrightarrow \mathbf{A}/\pi_L^{n+1}\mathbf{A}$ is $\pi_L^{n+1}\mathcal{O}_L$, i.e. we have exactness at the first position. Since we have $\varphi_L(x) = x$ for all $x \in \mathcal{O}_L$, we also have $\mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L \subseteq \ker(\mathrm{Fr} - \mathrm{id})$. So let $x \in \mathbf{A}$ such that $\mathrm{Fr}(x) - x \equiv 0 \bmod \pi_L^{n+1}\mathbf{A}$. Then we also have $\mathrm{Fr}(x) - x \equiv 0 \bmod \pi_L^n \mathbf{A}$ and because the first sequence is exact, we obtain a $y \in \mathcal{O}_L$ such that $y \equiv x \bmod \pi_L^n \mathbf{A}$. Then there is an $\alpha \in \mathbf{A}$ such that $x - y = \pi_L^n \alpha$, especially we have $x - y \equiv \pi_L^n \alpha \bmod \pi_L^{n+1}\mathbf{A}$. Since $\mathrm{Fr}(X) \equiv X^{q_L} \bmod \pi_L$ we get $\mathrm{Fr}(\alpha) \equiv \alpha^{q_L} \bmod \pi_L \mathbf{A}$ and therefore

$\mathrm{Fr}(\pi_L^n\alpha) \equiv \pi_L^n\alpha^{q_L} \bmod \pi_L^{n+1}\mathbf{A}$ since Fr is $\mathcal{O}_L$-linear. Then we also get

$$0 \equiv (\mathrm{Fr}-\mathrm{id})(x-y) \equiv (\mathrm{Fr}-\mathrm{id})(\pi_L^n\alpha) \equiv \pi_L^n(\alpha^{q_L}-\alpha) \bmod \pi_L^{n+1}\mathbf{A}.$$

Since $\mathbf{A}$ is a domain, this then implies $\alpha^{q_L} \equiv \alpha \bmod \pi_L\mathbf{A}$. Since the sequence in question is exact for $n=1$ by the start of the proof, we then get a $z \in \mathcal{O}_L$ such that $z \equiv \alpha \bmod \pi_L\mathbf{A}$, i.e. it exists $\beta \in \mathbf{A}$ such that $\alpha = z + \pi_L\beta$. We then get

$$x \equiv y + \pi_L^n\alpha \equiv y + \pi_L^n(z+\pi_L\beta) \equiv y + \pi_L^n z \bmod \pi_L^{n+1}\mathbf{A},$$

i.e. $\ker(\mathrm{Fr}-\mathrm{id}) \subseteq \mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L$.
It remains to check that $\mathrm{Fr}-\mathrm{id}$ is surjective on $\mathbf{A}/\pi_L^{n+1}\mathbf{A}$. So let $x \in \mathbf{A}$. Because $\mathrm{Fr}-\mathrm{id}$ is surjective on $\mathbf{A}/\pi_L^n\mathbf{A}$ we get a $y \in \mathbf{A}$ such that $\varphi_L(y)-y \equiv x \bmod \pi_L^n\mathbf{A}$. As before there is now an $\alpha \in \mathbf{A}$ such that $\varphi_L(y)-y \equiv x + \pi_L^n\alpha \bmod \pi_L^{n+1}\mathbf{A}$. Again, since the sequence for $n=1$ is exact we can find $z \in \mathbf{A}$ such that $\varphi_L(z)-z \equiv \alpha \bmod \pi_L\mathbf{A}$ and therefore we can find $\beta \in \mathbf{A}$ such that $\varphi_L(z)-z+\pi_L\beta = \alpha$. We then get

$$\begin{aligned}\mathrm{Fr}(y-\pi_L^n z)-(y-\pi_L^n z) &= \mathrm{Fr}(y)-y-\pi_L^n(\mathrm{Fr}(z)-z)\\ &\equiv x+\pi_L^n\alpha-\pi_L^n\alpha+\pi_L^{n+1}\beta \equiv x \bmod \pi_L^{n+1}\mathbf{A},\end{aligned}$$

i.e. $y-\pi_L^n z$ is $\bmod \pi_L^{n+1}\mathbf{A}$ a preimage of x under $\varphi_L-\mathrm{id}$.
Since the transition maps $\mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L \to \mathcal{O}_L/\pi_L^n\mathcal{O}_L$ are surjective, the inverse system $(\mathcal{O}_L/\pi_L^n\mathcal{O}_L)_n$ is a Mittag-Leffler System and therefore we have $\varprojlim^1 \mathcal{O}_L/\pi_L^n\mathcal{O}_L = 0$ (cf. Remark 2.3.9). By taking the inverse limit of the sequence

$$0 \longrightarrow \mathcal{O}_L/\pi_L^n\mathcal{O}_L \longrightarrow \mathbf{A}/\pi_L^n\mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\pi_L^n\mathbf{A} \longrightarrow 0$$

we then get the exact sequence

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

The proof of the exactness of the second sequence is similar to the prove above. Just replace $\mathbf{E}_L^{\mathrm{sep}}$ by $\mathbf{E}_L^{\mathrm{sep},+}$ which is the separable closure of $\mathbf{E}_L^+$ in $\mathbf{E}_L^{\mathrm{sep}}$.

2. As before we have $\mathrm{Fr}(x) = x^{q_L}$ for all $x \in \mathbf{E}_L^{\mathrm{sep}}$. Especially this equation holds for elements in $\mathbf{E}_L^{\mathrm{sep},+}$ and $\mathbf{E}_E^+$. The injectivity of the above maps then is easy

to see:
Let $0 \neq x \in \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$. So, in particular we have $\deg_{\omega_\phi}(x) \geq n > 0$ and therefore also $\deg_{\omega_\phi}(\mathrm{Fr}(x)) > \deg_{\omega_\phi}(x)$, i.e. $\mathrm{Fr}(x) - x \neq 0$ and so $\mathrm{Fr} - \mathrm{id}$ is injective on $\omega_\phi^n \mathbf{E}^+$. Because of $\mathbf{E}_E^+ \subseteq \mathbf{E}_L^{\mathrm{sep}}$ the homomorphism $\varphi_{E|L} - \mathrm{id}$ is also injective on $\omega_\phi^n \mathbf{E}_E^+$.
For the surjectivity let α be an element of $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ or of $\omega_\phi^n \mathbf{E}_E^+$. Then the series $(\mathrm{Fr}(\alpha)^i)_i$ converges to zero and therefore

$$\beta := \sum_{i=0}^{\infty} -\mathrm{Fr}(\alpha)^i$$

is also an element of $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ or of $\omega_\phi^n \mathbf{E}_E^+$ and clearly is a preimage of α under $\mathrm{Fr} - \mathrm{id}$.

3. Let $n, l \in \mathbb{N}$ be fixed and note that there is a canonical identification

$$(\omega_\phi^n \mathbf{A}^+)/(\pi_L^l \omega_\phi^n \mathbf{A}^+) \cong \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+)$$

since ω_ϕ^n is not a zero divisor in both $\mathbf{A}^+$ and $\mathbf{A}^+/\pi_L^l \mathbf{A}^+$. Now assume that

$$\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^k \mathbf{A}^+) \longrightarrow \omega_\phi^n(\mathbf{A}^+/\pi_L^k \mathbf{A}^+)$$

for all natural numbers $k \leq l$ is an isomorphism. Note that we just proved this for $l = 1$. Consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+) \arrow[r] & \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+) \\
\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+) \arrow[r] \arrow[u, two heads] & \omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+) \arrow[u, two heads]
\end{tikzcd}$$

Our aim is to show, that the latter horizontal homomorphism is also an isomorphism.
Let $x \in \mathbf{A}^+$ such that $\omega_\phi^n x \not\equiv 0 \bmod \pi_L^{l+1} \mathbf{A}^+$. The degree n-term (with respect to ω_ϕ) of $\mathrm{Fr}(\omega_\phi^n x) - \omega_\phi^n x$ is $\omega_\phi^n(\pi_L - 1)x$ and therefore it is unequal to zero modulo π_L^{l+1}. To see this, we assume $\omega_\phi^n(\pi_L - 1)x \equiv 0 \bmod \pi_L^{n+1}$ and let j be the smallest integer such that $2^j \geq n + 1$ and multiply this congruence with $(1 + \pi_L)(1 + \pi_L^2) \cdots (1 + \pi_L^{2^{j-1}})$. Then we get

$$0 \equiv \omega_\phi^n(\pi_L^j - 1)x \equiv -\omega_\phi^n x \bmod \pi_L^{n+1} \mathbf{A}^+$$

what we excluded, i.e. it has to be $\omega_\phi^n(\pi_L - 1)x \not\equiv 0 \bmod \pi_L^{n+1}\mathbf{A}^+$ and therefore $\mathrm{Fr} - \mathrm{id}$ is injective on $\omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1}\mathbf{A}^+)$.
Let $x \in \omega_\phi^n\mathbf{A}^+$. Then there exists $y \in \omega_\phi^n\mathbf{A}^+$ such that $\varphi_L(y) - y \equiv x \bmod \pi_L^l\mathbf{A}^+$ (because we assumed the surjectivity for all values $\leq l$), i.e. there exists $\alpha \in \omega_\phi^n\mathbf{A}^+$ such that $\mathrm{Fr}(y) - y = x + \pi_L^l\alpha$. Then again there exists $\beta \in \omega_\phi^n\mathbf{A}^+$ such that $\mathrm{Fr}(\beta) - \beta \equiv \alpha \bmod \pi_L$, i.e. there exists some $\eta \in \omega_\phi^n\mathbf{A}^+$ such that $\mathrm{Fr}(\beta) - \beta = \alpha + \pi_L\eta$. We then get

$$\begin{aligned}(\mathrm{Fr} - \mathrm{id})(y - \pi_L^l\beta) &= (\mathrm{Fr} - \mathrm{id})(y) - \pi_L^l(\mathrm{Fr} - \mathrm{id})(\beta) \\ &= x + \pi_L^l\alpha - \pi_L^l(\alpha + \pi_L\eta) \equiv x \bmod \pi_L^{l+1}\mathbf{A}^+,\end{aligned}$$

i.e. the map $\mathrm{Fr} - \mathrm{id}$ is surjective on $\omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1}\mathbf{A}^+)$. Since these maps are all isomorphisms, passing to the projective limit gives that the map $\mathrm{Fr} - \mathrm{id}$ is an isomorphism on $\omega_\phi^n\mathbf{A}^+$

□

Corollary 5.1.2.
For every $n \in \mathbb{N}$ the following sequence is exact:

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}/\omega_\phi^n\mathbf{A}^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\omega_\phi^n\mathbf{A}^+ \longrightarrow 0.$$

Proof.
In Lemma 5.1.1 we showed that

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

is an exact sequence and that

$$\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n\mathbf{A}^+ \longrightarrow \omega_\phi^n\mathbf{A}^+$$

is an isomorphism for every $n \in \mathbb{N}$. Since every element of the image of $\mathcal{O}_L \hookrightarrow \mathbf{A}$ has degree 0 (with respect to ω_ϕ) the homomorphism $\mathcal{O}_L \to \mathbf{A}/\omega_\phi^n\mathbf{A}^+$ is still injective. Since Fr fixes $\mathcal{O}_L$ it is clear that we have $\mathcal{O}_L \subseteq \ker(\mathrm{Fr} - \mathrm{id})$. For the other inclusion let $x \in \ker(\mathrm{Fr} - \mathrm{id})$. Then there exists an $\alpha \in \mathbf{A}$ such that $\alpha \bmod \omega_\phi^n\mathbf{A}^+ = x$ and $\mathrm{Fr}(\alpha) - \alpha \in \omega_\phi^n\mathbf{A}^+$. But since $\mathrm{Fr} - \mathrm{id}$ is an isomorphism on $\omega_\phi^n\mathbf{A}^+$ there exists also a $\beta \in \omega_\phi^n\mathbf{A}^+ \subseteq \mathbf{A}$ such that $\mathrm{Fr}(\beta) - \beta = \mathrm{Fr}(\alpha) - \alpha$. Because of the exactness of

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0$$

it then exists $\eta \in \mathcal{O}_L$ such that $\eta = \alpha - \beta$. This implies $\eta \equiv \alpha \bmod \omega_\phi^n \mathbf{A}^+$, i.e. $\eta = x$ which means $\ker(\mathrm{Fr} - \mathrm{id}) \subseteq \mathcal{O}_L$. This proves the exactness in the middle. For the surjectivity of $\mathrm{Fr} - \mathrm{id}$ recall that $\mathbf{A} \twoheadrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+$ and $\mathrm{Fr} - \mathrm{id} \colon \mathbf{A} \to \mathbf{A}$ are surjective and consider the commutative diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[r, two heads, "\mathrm{Fr}-\mathrm{id}"] \arrow[d, two heads] & \mathbf{A} \arrow[d, two heads] \\
\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \arrow[r, "\mathrm{Fr}-\mathrm{id}"'] & \mathbf{A}/\omega_\phi^n \mathbf{A}^+.
\end{tikzcd}$$

This implies that the homomorphism $\mathrm{Fr} - \mathrm{id} \colon \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \to \mathbf{A}/\omega_\phi^n \mathbf{A}^+$ is also surjective. □

Lemma 5.1.3.
Let $A|\mathbf{A}_L$ be a finite, unramified extension. Then, for every $m \in \mathbb{N}$, the canonical projection $A/\pi_L^{m+1} A \to A/\pi_L^m A$ has a continuous, set theoretical section with respect to the weak topology on A.

Proof.
From Proposition 3.5.4 we deduce that

$$A \cong \varprojlim_n \mathcal{O}_E/\pi_L^n \mathcal{O}_E((X))$$

for some finite, unramified extension $E|L$. Therefore we have

$$A/\pi_L^m A = \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X))$$

for every $m \in \mathbb{N}$. Therefore it is enough to give a continuous set theoretical section of the canonical projection $\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E((X)) \to \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X))$ with respect to the X-adic topology. Since the $\mathcal{O}_E/\pi_L^m \mathcal{O}_E$ are finite discrete, there exists for every $m \in \mathbb{N}$ a continuous map

$$\iota_m \colon \mathcal{O}_E/\pi_L^m \mathcal{O}_E \longrightarrow \mathcal{O}_E/\pi_L^{m+1} \mathcal{O}_E$$

which is a set theoretical section of the canonical projection. We then define a map

$$\begin{aligned}
\alpha_m \colon \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X)) &\longrightarrow \mathcal{O}_E/\pi_L^{m+1} \mathcal{O}_E((X)), \\
\textstyle\sum_{i>>-\infty} \lambda_i X^i &\longmapsto \textstyle\sum_{i>>-\infty} \iota_m(\lambda_i) X^i.
\end{aligned}$$

This then clearly is a set theoretical section of the canonical projection. We have to

check continuity.
So let $f \in \mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E((X))$ and $n \in \mathbb{N}_0$. If then $\alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$ is empty, there is nothing to prove. So assume there is $g \in \alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$ and let $h \in X^n\mathcal{O}_E/\pi_L^m\mathcal{O}_E[\![X]\!]$. Then g and $g+h$ coincide in degrees $< n$ and therefore, by definition, also $\alpha_m(g)$ and $\alpha_m(g+h)$ coincide in degrees $< n$, i.e.

$$\alpha_m(g+h) \in \alpha_m(g) + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!] = f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!]$$

since $\alpha_m(g) \in f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!]$. It then follows

$$g + X^n\mathcal{O}_E/\pi_L^m\mathcal{O}_E[\![X]\!] \subseteq \alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$$

and therefore that α_m is continuous. □

Corollary 5.1.4.
For every $m \in \mathbb{N}$ the canonical projection $\mathbf{A}/\pi_L^{m+1}\mathbf{A} \to \mathbf{A}/\pi_L^m\mathbf{A}$ has a continuous, set theoretical section.

Proof.
Since $\mathbf{A}$ is the π_L-adic completion of $\mathbf{A}_L^{\mathrm{nr}}$ it is

$$\mathbf{A}/\pi_L^m\mathbf{A} = \mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$$

for every $m \in \mathbb{N}$. Since colimits are exact it is

$$\mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}} = \bigcup_{A|\mathbf{A}_L \text{ fin, nr}} A/\pi_L^m A$$

for every $m \in \mathbb{N}$ and since we have for every $A|\mathbf{A}_L$ finite and unramified and every $m \in \mathbb{N}$ a continuous, set theoretical section of the canonical projection $A/\pi_L^{m+1}A \to A/\pi_L^m A$ (cf. Lemma 5.1.3) this induces for every $m \in \mathbb{N}$ a set theoretical section of the canonical projection $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^{m+1}\mathbf{A}_L^{\mathrm{nr}} \to \mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$, which then is continuous, since $\mathbf{A}_L^{\mathrm{nr}}$ carries the topology of the colimit and then so does $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$ for every $m \in \mathbb{N}$. □

Lemma 5.1.5.
Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$, set $M := \mathcal{M}_{K|L}(V)$ and $V_m := V/\pi_L^m V$ as well as $M_m := M/\pi_L^m M$ for $m \in \mathbb{N}$.
Then the transition maps of the inverse systems $(V_m)_m$, $(M_m)_m$ and $(\mathbf{A} \otimes_{\mathcal{O}_L} V_m)_m$ are surjective and they have a continuous, set theoretical section. In particular, the

short sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A} \otimes_{\mathcal{O}_L} V_m & \xrightarrow{\mathrm{id} \otimes \cdot \pi_L} & \mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} & \longrightarrow & \mathbf{A} \otimes_{\mathcal{O}_L} V_1 & \longrightarrow & 0, \\ 0 & \longrightarrow & M_m & \xrightarrow{\cdot \pi_L} & M_{m+1} & \longrightarrow & M_1 & \longrightarrow & 0 \end{array}$$

are exact and have continuous, set theoretical sections.

Proof.
Since $\mathcal{M}_{K|L}$ is exact as an equivalence of categories (cf. Theorem 3.9.1) and the tensor product is right exact, it is immediately clear that the transition maps of the systems $(M_m)_m$ and $(\mathbf{A} \otimes_{\mathcal{O}_L} V_m)_m$ are surjective since the transition maps of $(V_m)_m$ are.
Since the V_m are finite and discrete one can define a set theoretical section of the canonical projection $V_{m+1} \to V_m$ by choosing a preimage for every element in V_m. Since M_m is a finitely generated $\mathbf{A}_{K|L}$-module, there are for every $m \in \mathbb{N}$ isomorphisms of topological $\mathbf{A}_{K|L}$-modules

$$M_m \cong \bigoplus_{i=1}^{n^{(m)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m)}} \mathbf{A}_{K|L}$$

such that $n_i^{(m)} \leq n_{i+1}^{(m)}$ and the canonical projection $M_{m+1} \to M_m$ maps the i-th component of $\oplus_{i=1}^{n^{(m+1)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m+1)}} \mathbf{A}_{K|L}$ to the i-th component of $\oplus_{i=1}^{n^{(m)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m)}} \mathbf{A}_{K|L}$ for $i \geq n^{(m)}$ and is zero on the i-th component with $i > n^{(m)}$. With Lemma 5.1.3 we then obtain a continuous, set theoretical section for every component, which then also gives a continuous set theoretical section for $M_{m+1} \to M_m$.
As topological $\mathcal{O}_L$-module we have

$$\mathbf{A} \otimes_{\mathcal{O}_L} V_m \cong \bigoplus_{i=0}^{k^{(m)}} \mathbf{A}/\pi_L^{k_i^{(m)}} \mathbf{A}$$

and therefore we see that there exists a continuous, set theoretical section of the canonical projection $\mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} \to \mathbf{A} \otimes_{\mathcal{O}_L} V_m$ as above using Corollary 5.1.4 instead of Lemma 5.1.3.
The statement on the short exact sequences then follows immediately. □

Lemma 5.1.6.
Let $E|L$ be a finite extension and $H_E = \mathrm{Gal}(\overline{\mathbb{Q}_p}|E_\infty)$ as usual. Then the operation of H_E on $\mathbf{E}_L^{\mathrm{sep}}$ is continuous with respect to the discrete topology on $\mathbf{E}_L^{\mathrm{sep}}$.

Proof.

Let $x \in \mathbf{E}_L^{\mathrm{sep}}$. Then there exist a finite extension $\mathbb{F}|\mathbf{E}_E$ such that $x \in \mathbb{F}$. Then x is fixed by $U := \mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}}|\mathbb{F})$ which is an open subgroup of H_E. If then $\tau \in H$ and $y \in \mathbf{E}_L^{\mathrm{sep}}$ are such that $\tau(y) = x$, then $U\tau \times \{y\}$ is an open neighbourhood of $\{\tau\} \times \{y\}$ in $H_E \times \mathbf{E}_L^{\mathrm{sep}}$ with $\sigma(\tau(y)) = x$ for all $\sigma \in U$. □

Lemma 5.1.7.
Let V be a finite dimensional k_L-representation of G_K. Then there exists a finite Galois extension $E|K$ such that H_E acts trivially on V.

Proof.
Since the action of G_K on V is continuous, the homomorphism $G_K \to \mathrm{Aut}_{k_L}(V)$ is continuous and since V is a finite dimensional k_L-vector space, it is finite and so $\mathrm{Aut}_{k_L}(V)$ carries the discrete topology, i.e. the kernel of the upper homomorphism is open, which means that there exists a finite Galois extension $E|K$ such that G_E acts trivially on V. With G_E also H_E acts trivially on V. □

Lemma 5.1.8.
Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Then Δ acts on the short exact sequence

$$0 \to \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to 0$$

and it holds

1. *$H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ for all $j > 0$.*
2. *There exists $r \geq 0$ such that $\omega_\phi^r H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) = 0$ for all $j > 0$ and $n \in \mathbb{Z}$.*

Proof.
The proof is literally the same as the one of [Sch06, Lemma 2.2.10, p.20] □

Lemma 5.1.9.
Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Then we have

1. *$(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K} \cong (\mathbf{E}_E \otimes_{k_L} V)^\Delta$.*
2. *$(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \cong (\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta$ for all $n \geq 0$.*

Proof.
In both cases the proof is the same. So let X be $\mathbf{E}_L^{\mathrm{sep}}$ or $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ for some $n \geq 0$. Note that $H_K/H_E \cong \Delta$. We then get

$$(X \otimes_{k_L} V)^{H_K} = ((X \otimes_{k_L} V)^{H_E})^{H_K/H_E} = (X^{H_E} \otimes_{k_L} V)^{\Delta},$$

where the last equation is true, since H_E acts trivial on V. □

Before stating a corollary, we should introduce some notation. Since all projective systems which appear here are indexed by the natural numbers, we will make the following definitions only for projective systems indexed by natural numbers.

Proposition 5.1.10.
Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Let in addition $M = \mathcal{M}_{K|L}(V)$ and

$$M_n := M/\left(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V\right)^{H_K}.$$

Then we have

1. *The inverse systems $(H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ and $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ are ML-zero for all $j > 0$.*

2. *The map of inverse systems $(M_n)_n \to (H^0(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is an ML-isomorphism.*

Proof.

1. Since V is a finite dimensional k_L-vector space, it's flat and therefore the homomorphism $\omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V \subseteq \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V$ is injective and induces a homomorphism

$$H^j(\Delta, \omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V).$$

The image of this last homomorphism is a subset of $\omega_\phi H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$, i.e. the maps $H^j(\Delta, \omega_\phi^k \mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ are zero for $k \geq n + r$ (cf. Lemma 5.1.8, 2.), i.e. the inverse system $(H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero for $j > 0$.
Since every class in $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+$ has a unique representative of highest degree $\leq n - 1$ in ω_ϕ the homomorphism $\mathbf{E}_E \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+$ has a set theoretical

splitting (by sending a class to this representative). This map is continuous, since the preimage of a subset of $\mathbf{E}_E$ in $\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+$ is equal to the image under the canonical projection, which is open by definition. Since V is flat, the sequence

$$0 \to \omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V \to 0$$

is exact and we can deduce a long exact cohomology sequence (cf. [NSW08, (2.3.2) Lemma, p.106]) and since $H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ for $j > 0$ (cf. Lemma 5.1.8, 1.), the homomorphism

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V) \to H^{j+1}(\Delta, \omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)$$

is an isomorphism for all $j > 0$ and the diagram

$$\begin{tikzcd}
H^j(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V) \arrow[r] & H^{j+1}(\Delta, \omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V) \\
H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n+1}\mathbf{E}_E^+ \otimes_{k_L} V) \arrow[r] \arrow[u] & H^{j+1}(\Delta, \omega_\phi^{n+1}\mathbf{E}_E^+ \otimes_{k_L} V) \arrow[u]
\end{tikzcd}$$

commutes. This means that the transition map

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^k\mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)$$

is zero for $k \geq n+r$ and therefore the inverse system $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero.

2. As seen before, for every $n \geq 0$ we have an exact sequence

$$0 \to \omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V \to 0.$$

Taking Δ-invariants then gives an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & (\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \arrow[r] & (\mathbf{E}_E \otimes_{k_L} V)^\Delta \arrow[r] & \cdots \\
\cdots \arrow[r] & (\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \arrow[r] & H^1(\Delta, \omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V) \arrow[r] & 0,
\end{tikzcd}$$

where the last term is zero because $H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ for $j > 0$ (cf. Lemma

5.1.8, 1.). With Lemma 5.1.9 this sequences becomes

$$\begin{aligned} 0 \longrightarrow (\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \longrightarrow (\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K} \longrightarrow \cdots \\ \cdots \longrightarrow (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow 0 \end{aligned}$$

and then gives the following short exact sequence

$$\begin{aligned} 0 \longrightarrow (\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K}/(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \longrightarrow \cdots \\ \cdots \longrightarrow (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow 0. \end{aligned}$$

In particular, $H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ is the cokernel of the homomorphism $M_n \to (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta$. According to the first part of the proof the inverse system $(H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero, and since the kernel of $M_n \to (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes V)^\Delta$ is zero it is also ML-zero, which then ends the proof.

□

Theorem 5.1.11.
Let $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and set $M = \mathcal{M}_{K|L}(V)$. Then there are isomorphisms

$$H^*_{\mathrm{cts}}(G_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M),$$

$$H^*_{\mathrm{cts}}(H_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(M).$$

These isomorphisms are functorial in V and compatible with restriction and corestriction.

Proof. In this proof, we follow the proof of [Sch06, Theorem 2.2.1, p.702ff]

Step 1: Explaining the strategy.
First, for $m \in \mathbb{N}$ set $V_m := V/\pi_L^m V$ and $M_m := M/\pi_L^m M$. Since $\mathcal{M}_{K|L}$ is an equivalence of categories (cf. Theorem 3.9.1) it is exact and therefore we have $M_m = \mathcal{M}_{K|L}(V_m)$. The open subgroups

$$M_m \cap \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m\right) = \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m\right)^{H_K}$$

form a basis of neighbourhoods of 0 in M_m. These subgroups are clearly stable under the operation of Γ_K and since $\varphi_{K|L}$ commutes with the operation of G_K on

$\left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m\right)$ these subgroups are also stable under $\varphi_{K|L}$. We then set

$$M_{m,n} := M_m / \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m\right)^{H_K}.$$

Since $\left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m\right)^{H_K}$ is an open subgroup, this is a discrete Γ_K-module and we have topological isomorphisms

$$M_m \cong \varprojlim_n M_{m,n}$$
$$M \cong \varprojlim_m M_m.$$

In Corollary 5.1.2 we proved that the sequence

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \longrightarrow 0$$

is exact and since $\mathbf{A}/\omega_\phi^n \mathbf{A}^+$ is a free $\mathcal{O}_L$-module, it is flat and therefore the sequence

$$0 \longrightarrow V_m \longrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \longrightarrow 0$$

is also exact. Then Lemma 2.3.3 says that for every $m, n \geq 1$ we have a quasi isomorphism

$$C^\bullet_{\mathrm{cts}}(G_K, V_m) \longrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m).$$

The inverse systems $(V_m)_m$ and $((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m)_{n,m}$ have surjective transition maps. From Corollary 2.1.12 we then can deduce that also the inverse systems of complexes $(C^\bullet_{\mathrm{cts}}(G_K, V_m))_m$ and $C^\bullet_{\mathrm{cts}}(G_K, ((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m))_{n,m}$ have surjective transition maps and Lemma 2.3.8 then says that the system $\mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, ((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m))_{n,m}$ has surjective transition maps as well.
From the quasi isomorphism $C^\bullet_{\mathrm{cts}}(G_K, V_m) \to \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes V_m))$ we then can deduce with Proposition 2.3.11 that the cohomologies of the complexes $\varprojlim_{n,m} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$ and $\varprojlim_m C^\bullet_{\mathrm{cts}}(G_K, V_m)$ coincide. Since $\varprojlim_m C^\bullet_{\mathrm{cts}}(G_K, V_m) \cong C^\bullet_{\mathrm{cts}}(G_K, V)$, the cohomology of $\varprojlim_m C^\bullet_{\mathrm{cts}}(G_K, V_m)$ is $H^*_{\mathrm{cts}}(G_K, V)$, which then is also computed by $\varprojlim_{n,m} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$.
On the other hand, since the canonical inclusion $\iota \colon M_{m,n} \hookrightarrow (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m$ commutes with $\varphi_{K|L}$ and since together with the canonical projection $\mathrm{pr} \colon G_K \twoheadrightarrow \Gamma_K$ it holds

$$\iota(\mathrm{pr}(\sigma)x) = \sigma\iota(x)$$

for all $\sigma \in G_K$ and $x \in M_{m,n}$ and since the operations of $\varphi_{K|L}$ and G_K respectively Γ_K commute wo get an induced morphism of complexes

$$\alpha_{m,n} \colon \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_{m,n}) \to \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m)$$

(cf. [NSW15, I §5, p45], the additional properties concerning $\varphi_{K|L}$ we noted above, ensure that we get the morphism of the above total complex with respect to $\varphi_{K|L}$ on the left hand side and Fr on the right hand side).
We now want to see that $\varprojlim_{n,m} \alpha_{m,n}$ is a quasi isomorphism. Because of $\varprojlim_{n,m} \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_{m,n}) = \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M)$ (cf. Lemma 2.3.7), this then says that the cohomology of $\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M)$ and $\varprojlim_{n,m} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$ coincide. But then the cohomologies of $\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M)$ and $C^\bullet_{\mathrm{cts}}(G_K, V)$ coincide, what is exactly what we want to prove.
To see that $\varprojlim_{n,m} \alpha_{m,n}$ is a quasi isomorphism, it is enough to see, that $\varprojlim_n \alpha_{m,n}$ is a quasi isomorphism for every $m \geq 1$. Because if this is shown, one knows that the inverse systems of complexes $(\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_m))_m$ and $(\mathcal{C}^\bullet_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m))_m$ are quasi isomorphic. Since the transition maps $M_{m+1} \to M_m$ as well as $\mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} \to \mathbf{A} \otimes_{\mathcal{O}_L} V_m$ are surjective and have a continuous section (cf. Lemma 5.1.5), one can see as before, using Corollary 2.1.12 and Lemma 2.3.8, that the inverse systems of complexes $(\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_m))_m$ and $(\mathcal{C}^\bullet_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m))_m$ have surjective transition maps. As before with Proposition 2.3.11 respectively Remark 2.3.12 one then sees that $\varprojlim_m \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_m)$ and $\varprojlim_m \mathcal{C}^\bullet_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m)$ are quasi isomorphic.
So, what is still to show, is that $\varprojlim_n \alpha_{m,n}$ is a quasi isomorphism for every $m \geq 1$. This will be the rest of the proof.

Step 2: Reduction to the case $m = 1$.
Since for every $m \geq 1$ the sequence

$$0 \longrightarrow V_m \longrightarrow V_{m+1} \longrightarrow V_1 \longrightarrow 0.$$

is exact and $\mathcal{M}_{K|L}$ is an exact functor (since it is an equivalence), this implies that for every $m \geq 1$ there is a short exact sequence

$$0 \longrightarrow M_m \longrightarrow M_{m+1} \longrightarrow M_1 \longrightarrow 0.$$

By the definition of the topology on the M_m it is clear, that the topology of M_m is induced from that of M_{m+1} and from Lemma 5.1.5 we deduce that it has a continuous set theoretical section. Therefore Proposition 2.2.35 says that we get a long exact sequence of cohomology.

Now assume the result is shown for $m = 1$. Then $\mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M) \to H^*_{\mathrm{cts}}(G_K, V)$ is an isomorphism for every V with $\pi_L V = 0$. Induction on m and the 5-lemma applied to the following diagram which arises from the long exact cohomology sequences (where we write $\Gamma = \Gamma_K$ and $G = G_K$ and $\varphi = \varphi_{K|L}$)

$$\begin{tikzcd}
\mathcal{H}^{l-1}_{\varphi}(\Gamma, M_1) \arrow[r, "\delta"] \arrow[d, "\cong"'] & \mathcal{H}^{l}_{\varphi}(\Gamma, M_m) \arrow[r] \arrow[d, "\cong"'] & \mathcal{H}^{l}_{\varphi}(\Gamma, M_{m+1}) \arrow[r] \arrow[d] & \mathcal{H}^{l}_{\varphi}(\Gamma, M_1) \arrow[r, "\delta"] \arrow[d, "\cong"'] & \mathcal{H}^{l+1}_{\varphi}(\Gamma, M_m) \arrow[d, "\cong"'] \\
H^{l-1}_{\mathrm{cts}}(G, V_1) \arrow[r, "\delta"] & H^{l}_{\mathrm{cts}}(G, V_m) \arrow[r] & H^{l}_{\mathrm{cts}}(G, V_{m+1}) \arrow[r] & H^{l}_{\mathrm{cts}}(G, V_1) \arrow[r, "\delta"] & H^{l+1}_{\mathrm{cts}}(G, V_m)
\end{tikzcd}$$

then implies the result for all $m \geq 1$.

Step 3: Splitting $\alpha_{1,n}$ up.

For the rest of the proof we may assume $\pi_L V = 0$ and therefore also $\pi_L M = 0$, but we will still write $M_{1,n}$ to avoid confusion. Note that this implies

$$\mathbf{A} \otimes_{\mathcal{O}_L} V \cong \mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V,$$

$$\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V \cong \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$$

as well as the correspondingly isomorphism with respect to the fixed modules of H_K.

Now fix a finite Galois extension $E|K$ such that H_E acts trivially on V (cf. Lemma 5.1.7). Then, the canonical inclusion

$$M_{1,n} = \frac{(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K}}{(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K}} \hookrightarrow \frac{(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_E}}{(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_E}} = \mathbf{E}_E / \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V$$

induces together with the canonical projection $\mathrm{Gal}(E_\infty|K) \twoheadrightarrow \Gamma_K$, as in step 1 for $\alpha_{m,n}$, for all $n \in \mathbb{N}$ a morphism of complexes

$$\beta_n \colon \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_{1,n}) \longrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V).$$

Simultaneously, the canonical inclusion $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$ together with the canonical projection $G_K \twoheadrightarrow \mathrm{Gal}(E_\infty|K)$ induces for all $n \in \mathbb{N}$ a morphism of complexes

$$\gamma_n \colon \mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V).$$

Since both diagramms

$$\begin{tikzcd} M_{1,n} \arrow[r, hook] \arrow[dr, hook] & \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{kL} V \arrow[d, hook] \\ & \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V, \end{tikzcd} \qquad \begin{tikzcd} \Gamma_K & \mathrm{Gal}(E_\infty|K) \arrow[l, two heads] \\ & G_K \arrow[ul, two heads] \arrow[u, two heads] \end{tikzcd}$$

are commutative, where all the arrows in the left diagram are canonical inclusions and the ones in the right diagram are canonical projections, it is immediately clear that also the diagramm

$$\begin{tikzcd} \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_{1,n}) \arrow[r, "\beta_n"] \arrow[dr, "\alpha_{1,n}"'] & \mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V) \arrow[d, "\gamma_n"] \\ & \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V) \end{tikzcd}$$

commutes. So, to prove that $\varprojlim_n \alpha_{1,n}$ is a quasi-isomorphism it is enough to prove that $\varprojlim_n \beta_n$ and $\varprojlim_n \gamma_n$ are quasi-isomorphisms. In addition, we will also show that γ_n is a quasi-isomorphism for every $n \geq 1$.

Step 4: $\varprojlim_n \gamma_n$ is a quasi-isomorphism.
Due to Lemma 2.2.34 there is an E_2-spectral sequence converging to the cohomology of the source of γ_n

$$H^a(\mathrm{Gal}(E_\infty|K), \mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)) \Longrightarrow \\ \mathcal{H}^{a+b}_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)$$

as well as en E_2-spectral sequence converging to the target of γ_n

$$H^a(\mathrm{Gal}(E_\infty|K), \mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)) \Longrightarrow \\ \mathcal{H}^{a+b}_{\mathrm{Fr}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V).$$

The canonical inclusion $\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$ together with the trivial map $H_E \to 1$ then induces a homomorphism on the above E_2-pages. Together with the from γ_n induced map on cohomology this then gives a morphism of spectral sequences. So, to show that γ_n induces an isomorphism on cohomology it is enough to show that the induced homomorphism on the above E_2 pages is an isomorphism. And for this it is enough, that the homomorphism between the coefficients $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)$ and $\mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)$ is an isomorphism. Since H_E acts trivially on V it is (cf. Proposition 2.3.14)

$$\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+\otimes_{k_L}V)=\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+)\otimes_{k_L}V$$
$$\mathcal{H}^b_{\mathrm{Fr}}(H_E,\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+}\otimes_{k_L}V)=\mathcal{H}^b_{\mathrm{Fr}}(H_E,\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+})\otimes_{k_L}V.$$

Therefore it is enough to show that there is an isomorphism between $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+)$ and $\mathcal{H}^b_{\mathrm{Fr}}(H_E,\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+})$. To see this, consider the commutative square

$$\begin{tikzcd}
\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E) \arrow[r] \arrow[d] & \mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+) \arrow[d] \\
\mathcal{H}^b_{\mathrm{Fr}}(H_E,\mathbf{E}_L^{\mathrm{sep}}) \arrow[r] & \mathcal{H}^b_{\mathrm{Fr}}(H_E,\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+}),
\end{tikzcd}$$

where $\mathbf{E}_L^{\mathrm{sep}}$ is regarded as discrete H_E-module (cf. Lemma 5.1.6) and where the horizontal maps are induced from the respective canonical projections and the vertical maps from the respective canonical inclusions.

First we want to see, that the upper horizontal map is an isomorphism. $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E)$ is computed by $\mathbf{E}_E\xrightarrow{\varphi_L-\mathrm{id}}\mathbf{E}_E$ and $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+)$ by the corresponding complex and the square

$$\begin{tikzcd}
\mathbf{E}_E \arrow[r, "\mathrm{Fr}-\mathrm{id}"] \arrow[d] & \mathbf{E}_E \arrow[d] \\
\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \arrow[r, "\mathrm{Fr}-\mathrm{id}"] & \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+
\end{tikzcd}$$

is commutative. Denote the kernel and image of the upper horizontal map by κ_1 and im_1 and the ones of the lower vertical map by κ_2 and im_2 respectively. By Lemma 5.1.1 the map $\omega_\phi^n\mathbf{E}_E^+\xrightarrow{\mathrm{Fr}-\mathrm{id}}\omega_\phi^n\mathbf{E}_E^+$ is an isomorphism, especially is $\omega_\phi^n\mathbf{E}_E^+\subseteq\mathrm{im}_1$ and so we see immediately $\mathrm{im}_2\subseteq\mathrm{im}_1/\omega_\phi^n\mathbf{E}_E^+$. For the other inclusion let $\overline{x}\in\mathrm{im}_1/\omega_\phi^n\mathbf{E}_E^+$ and $x\in\mathbf{E}_E$ a preimage under the canonical projection. Because of $\omega_\phi^n\mathbf{E}_E^+\subseteq\mathrm{im}_1$ we deduce $x\in\mathrm{im}_1$. If $y\in\mathbf{E}_E$ is a preimage of x under $\mathrm{Fr}-\mathrm{id}$, then because of the commutativity of the latter diagram we get $(\mathrm{Fr}-\mathrm{id})(\overline{y})=\overline{x}$, i.e. $\overline{x}\in\mathrm{im}_2$. Therefore $\mathcal{H}^1_{\mathrm{Fr}}(\mathbf{E}_E)$ and $\mathcal{H}^1_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+)$ coincide.

For the term in degree zero let $x\in\kappa_1$ such that $x\in\omega_\phi^n\mathbf{E}_E^+$. Since $\mathrm{Fr}-1$ is an isomorphism on $\omega_\phi^n\mathbf{E}_E^+$ and $(\mathrm{Fr}-1)(x)=0$, x itself is zero, i.e. the canonical homomorphism $\kappa_1\to\kappa_2$ is injective. Let now $\eta\in\kappa_2$ and $y'\in\mathbf{E}_E$ a preimage under the canonical projection. By commutativity it is $\overline{(\mathrm{Fr}-1)(y')}=0$ and therefore $(\mathrm{Fr}-1)(y')\in\omega_\phi^n\mathbf{E}_E^+$. Again since $\mathrm{Fr}-1$ is an isomorphism on $\omega_\phi^n\mathbf{E}_E^+$ we find an element $y''\in\omega_\phi^n\mathbf{E}_E^+$ with $(\mathrm{Fr}-1)(y')=(\mathrm{Fr}-1)(y'')$. Set $y:=y'-y''$. Then $\overline{y}=\overline{y'}-\overline{y''}=\overline{y'}=\eta$ and $(\mathrm{Fr}-1)(y)=0$, i.e. $\kappa_1\to\kappa_2$ is also surjective and therefore an isomorphism. Since every other cohomology group is zero, we conclude that

$$\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E) \cong \mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+)$$

for all $b \geq 0$.

For the lower horizontal map in the upper square, recall that Lemma 5.1.1 also says that $\mathrm{Fr}-1$ is on $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ an isomorphism. Therefore one sees with a similar argument as above that the canonical projection $\mathbf{E}_L^{\mathrm{sep}} \twoheadrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ induces an isomorphism between the cohomology groups $\mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}_L^{\mathrm{sep}})$ and $\mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+})$ for all $b' \geq 0$. Lemma 2.2.24 states that there are two E_2-spectral sequences converging to $\mathcal{H}^*_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}})$ respectively $\mathcal{H}^*_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+})$ (recall from the beginning of Step 4 that $\mathbf{E}_L^{\mathrm{sep}}$ is considered as discrete H_E-module):

$$\begin{aligned} H^{a'}(H_E, \mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}_L^{\mathrm{sep}})) &\Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}) \\ H^{a'}(H_E, \mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+})) &\Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}). \end{aligned}$$

We conclude as before: The canonical projection $\mathbf{E}_L^{\mathrm{sep}} \twoheadrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ induces a morphism of spectral sequences and since the induced homomorphism is an isomorphism on the E_2-pages, we obtain an isomorphism between the limit terms $\mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}})$ and $\mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+})$ for all $b \geq 0$.

To see that the left vertical arrow in the first square is an isomorphism we consider the E_2-spectral sequence (cf. Lemma 2.2.24)

$$\mathcal{H}^{a'}_{\mathrm{Fr}}(H^{b'}(H_E, \mathbf{E}_L^{\mathrm{sep}})) \Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}}).$$

Since $\mathbf{E}_L^{\mathrm{sep}}$ is a separabel closure of $\mathbf{E}_E$ with Galois group isomorphic to H_E it ist $H^{b'}(H_E, \mathbf{E}_L^{\mathrm{sep}}) = 0$ for all $b' > 0$. Then [NSW15, Chapter II §1, (2.1.4) Proposition, p.100] says that we have an isomorphism $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E) \cong \mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}_L^{\mathrm{sep}})$ for all $b \geq 0$ (here we identified $H^0(H_E, \mathbf{E}_L^{\mathrm{sep}}) = (\mathbf{E}_L^{\mathrm{sep}})^{H_E} = \mathbf{E}_E$), which is induced from the canonical inclusion, i.e. the left vertical arrow in the first square also is an isomorphism. Then also the right vertical arrow is an isomorphism (since all other arrows are isomorphims) and so is the map on E_2-terms from which we started. Hence γ_n is a quasi-isomorphism for all n.

To see that $\varprojlim_n \gamma_n$ is an isomorphism, it remains to check that the transition maps are surjective (cf. Proposition 2.3.11 respectively Remark 2.3.12). Since the transition maps

$$\mathbf{E}_E/\omega_\phi^{n+1}\mathbf{E}_E^+ \otimes_{k_L} V \twoheadrightarrow \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V,$$
$$\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n+1}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V \twoheadrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$$

are surjective and the groups carry the discrete topology, Corollary 2.1.12 says that also the transition maps

$$C^\bullet_{\mathrm{cts}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^{n+1}\mathbf{E}_E^+ \otimes_{k_L} V) \twoheadrightarrow C^\bullet_{\mathrm{cts}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V),$$
$$C^\bullet_{\mathrm{cts}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n+1}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V) \twoheadrightarrow C^\bullet_{\mathrm{cts}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)$$

are surjective. But then Lemma 2.3.8 says that the transition maps

$$\mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^{n+1}\mathbf{E}_E^+ \otimes_{k_L} V) \twoheadrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V),$$
$$\mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n+1}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V) \twoheadrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^{n}\mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)$$

are surjective, too. Then Proposition 2.3.11 respectively Remark 2.3.12 say that $\varprojlim_n \gamma_n$ is a quasi isomorphism.

<u>**Step 5:**</u> $\varprojlim_n \beta_n$ is a quasi-isomorphism.

Now let $\Delta := \mathrm{Gal}(E_\infty \mid K_\infty)$. Lemma 2.2.34 then says that there is an E_2-spectral sequence of inverse systems of abelian groups given by

$$\mathcal{H}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V)) \Longrightarrow \mathcal{H}^{i+j}_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V).$$

We will write ${}_n\mathscr{E}_2^{ij}$ for second page of this E_2-spectral sequence, ${}_n\mathscr{E}^k$ for its limit term and $\mathscr{E}_2^{ij} = \varprojlim_n {}_n\mathscr{E}_2^{ij}$ as wells as $\mathscr{E}^k = \varprojlim_n {}_n\mathscr{E}^k$. Proposition 5.1.10 says that the system $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero for $j > 0$, i.e. for every $n \in \mathbb{N}$ there is an $m(n) \in \mathbb{N}$ such that the transition map

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)}\mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V)$$

is the zero map. For fixed $n \in \mathbb{N}$ and $m(n) \in \mathbb{N}$ as above, we then obtain that the transition map

$$C^i_{\mathrm{cts}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)}\mathbf{E}_E^+ \otimes_{k_L} V)) \longrightarrow C^i_{\mathrm{cts}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V))$$

is also zero for all $i \geq 0$ and $j > 0$. Then clearly the transition map

$$\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)}\mathbf{E}_E^+ \otimes_{k_L} V)) \longrightarrow \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n}\mathbf{E}_E^+ \otimes_{k_L} V))$$

is zero for all $i \geq 0$ and $j > 0$, too. And so is the induced map on cohomology, i.e the inverse systems $({}_n\mathscr{E}_2^{ij})_n$ are ML-zero for all $i \geq 0$ and $j > 0$. But then the edge homomorphism $\mathscr{E}_2^{i0} \to \mathscr{E}^i$ is an isomorphism, since $\mathscr{E}_2^{ij} = 0$ for all $i \geq 0$ and $j > 0$ (cf. [NSW15, Chapter II, §1, (2.1.4) Corollary, p.100]). Recall that this edge homomorphism is induced from both, the canonical projection $\mathrm{Gal}(E_\infty|K) \twoheadrightarrow \Gamma_K$ and the canonical inclusion $(\mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \hookrightarrow \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V$.
Proposition 5.1.10 says that $(\eta_n)_n \colon (M_{1,n})_n \to (H^0(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V))_n$ is an ML-isomorphism. Therefore the inverse systems $(\ker(\eta_n))_n$ and $(\mathrm{coker}(\eta_n))_n$ are ML-zero. As above, we then deduce that also the systems $(\mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, \ker(\eta_n)))_n$ and $(\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \mathrm{coker}(\eta_n)))_n$ are ML-zero for all $i \in \mathbb{N}_0$. Since $H^0(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)$ and $M_{1,n}$ carry the discrete topology for all $n \in \mathbb{N}$, we deduce from Lemma 2.2.26, which says that $\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, -)$ is for discrete modules an exact functor, the exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, \ker(\eta_n)) \longrightarrow \mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, M_{1,n}) \xrightarrow{\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \eta_n)} \cdots \\ \cdots \longrightarrow \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E/\omega_\phi^n\mathbf{E}_E^+ \otimes_{k_L} V)) \longrightarrow \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \mathrm{coker}(\eta_n)) \longrightarrow 0. \end{array}$$

Taking inverse limits then gives us an isomorphism of complexes

$$\mathfrak{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_1) \cong \mathcal{C}^\bullet_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E \otimes_{k_L} V)),$$

which, by construction, is induced from the canonical inclusion $M_1 \hookrightarrow \mathbf{E}_E \otimes_{k_L} V$ and which then prolongs to an isomorphism of its respective cohomology groups, i.e. for all $i \in \mathbb{N}_0$ we get

$$\mathcal{H}^i_{\varphi_{K|L}}(\Gamma_K, M_1) \cong \mathcal{H}^i_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E \otimes_{k_L} V)).$$

Together with the observation from above, that the edge homomoprhism $\mathscr{E}_2^{i0} \to \mathscr{E}^i$ is an isomorphism for all $i \in \mathbb{N}_0$ we deduce for all $i \in \mathbb{N}_0$ the isomorphism

$$\mathcal{H}^i_{\varphi_{K|L}}(\Gamma_K, M_1) \cong \mathcal{H}^i_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E \otimes_{k_L} V),$$

which by construction is $\varprojlim_n \beta_n$.

$\square$

5.2 Description with ψ

In this section we want to give a description of the Galois cohomology groups of a representation using a ψ-operator.

Definition 5.2.1.
Let A be an $\mathcal{O}_L$-module. We say that A is **cofinitely generated** if its Pontrjagin dual $A^\vee = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(A, L/\mathcal{O}_L)$ is finitely generated.

Remark 5.2.2.

1. *Since finitely generated $\mathcal{O}_L$-modules together with their natural topology are compact, cofinitely generated $\mathcal{O}_L$-modules are discrete, which means that $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(-, L/\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_L}(-, L/\mathcal{O}_L)$ for both, finitely and cofinitely generated $\mathcal{O}_L$-modules.*

2. *For $n \in \mathbb{N}$ we have an isomorphism*

$$\begin{aligned} \mathcal{O}_L/\pi_L^n\mathcal{O}_L &\longrightarrow (\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\vee \\ x \bmod \pi_L^n\mathcal{O}_L &\longmapsto [1 \bmod \pi_L^n\mathcal{O}_L \mapsto \pi_L^{-n}x \bmod \mathcal{O}_L] \end{aligned}$$

   which then also implies a non-canocial isomorphism $T \cong T^\vee$ for a finitely generated torsion $\mathcal{O}_L$-module, since $(-)^\vee$ is compatible with finite direct sums. These isomorphisms are clearly topological, since all these objects carry the discrete topology.

3. *Due to Pontrjagin duality (cf. Proposition 4.3.2) a cofinitely generated $\mathcal{O}_L$-module is always the Pontrjagin dual of a finitely generated $\mathcal{O}_L$-module.*

4. *If $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ is torsion, then $T^\vee$ also is a finitely generated torsion $\mathcal{O}_L$-module with a continuous action from G_K.*

Definition 5.2.3.
Let A be a cofinitely generated $\mathcal{O}_L$-module and $n \in \mathbb{N}$. We denote by A_n the kernel of the multiplication $\mu_{\pi_L^n}$ with π_L^n on A, i.e.

$$A_n = \ker(\mu_{\pi_L^n} \colon A \to A).$$

Proposition 5.2.4.
Let A be a cofinitely generated $\mathcal{O}_L$-module. Then we have $A = \varinjlim_n A_n$.
In particular, if A is torsion, say with $\pi_L^m A = 0$ for some $m \in \mathbb{N}$, then we have $A = A_m$.

Proof.
Let T be a finitely generated $\mathcal{O}_L$-module such that $A = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(T, L/\mathcal{O}_L)$, let $e_1, \ldots, e_m$ be a set of generators of T and let $f \in A$. Then for every $i \in \{1, \ldots, m\}$ there exists an $n_i \in \mathbb{N}$ such that $\pi_L^{n_i} f(e_i) = 0$. Set $n := \max_i n_i$. Then it is $\pi_L^n f(\alpha) = 0$ for every $\alpha \in A$, i.e. $f \in A_n$.
In particular, if there exists $m \in \mathbb{N}$ such that $\pi_L^m g = 0$ for every $g \in A$, then the above shows $A = A_m$. □

Lemma 5.2.5.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ such that $\pi_L^m T = 0$. Then H_K acts continuously on $\mathbf{A} \otimes_{\mathcal{O}_L} T$ equipped with the discrete topology.

Proof.
Recall from page 57 that

$$\mathbf{A} \cong \varprojlim_n \mathbf{A}_L^{\mathrm{nr}} / \pi_L^n \mathbf{A}_L^{\mathrm{nr}}$$

and that H_L is the Galois group of $\mathbf{A}_L^{\mathrm{nr}} | \mathbf{A}_L$. The latter means, that H_L acts continuously on $\mathbf{A}_L^{\mathrm{nr}}$ with respect to the discrete topology because if $x \in \mathbf{A}_L^{\mathrm{nr}}$, then $\mathbf{B}_L(x) | \mathbf{B}_L$ is a finite extension and therefore it exists an open subgroup $U \leq H_L$ which fixes x. But then (U, x) is an open subset of the preimage of x under the operation

$$H_L \times \mathbf{B}_L \to \mathbf{B}_L.$$

Then H_L also clearly acts continuously on $\mathbf{A}_L^{\mathrm{nr}} / \pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ for all $n \in \mathbb{N}$ equipped with the discrete topology. Since H_K is an open subgroup of H_L it then also acts continuously on $\mathbf{A}_L^{\mathrm{nr}} / \pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ for all $n \in \mathbb{N}$ equipped with the discrete topology. Because of $\pi_L^n T = 0$ we have $T = T \otimes_{\mathcal{O}_L} \mathcal{O}_L / \pi_L^n \mathcal{O}_L$ and therefore

$$\mathbf{A} \otimes_{\mathcal{O}_L} T = \mathbf{A} \otimes_{\mathcal{O}_L} \mathcal{O}_L / \pi_L^n \mathcal{O}_L \otimes_{\mathcal{O}_L} T = \mathbf{A} / \pi_L^n \mathbf{A} \otimes_{\mathcal{O}_L} T = \mathbf{A}_L^{\mathrm{nr}} / \pi_L^n \mathbf{A}_L^{\mathrm{nr}} \otimes_{\mathcal{O}_L} T.$$

Since H_K acts continuously on both T and $\mathbf{A}_L^{\mathrm{nr}} / \pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ with respect to the discrete topology it does so on the tensor product equipped with the linear topological structure, which then again is discrete. □

Lemma 5.2.6.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ such that $\pi_L^m T = 0$. Then we have $H^i_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} T) = 0$ for all $i > 0$.

Proof.
This is [SV15, Lemma 5.2, p. 23–24], since it is even $H^i_{\mathrm{cts}}(U, \mathbf{E}_L^{\mathrm{sep}}) = 0$ for all $i > 0$

and open subgroups $U \leq H_L$. $\square$

Corollary 5.2.7.
Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then H_K acts continuously on $\mathbf{A} \otimes_{\mathcal{O}_L} A$ equipped with the discrete topology and we have $H^i_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ for all $i > 0$.

Proof.
If A is torsion, then Remark 5.2.2 says that this is just Lemma 5.2.5 and Lemma 5.2.6.
If A is general, then with Proposition 5.2.4 we can write $A = \varinjlim_n A_n$, where the A_n are torsion $\mathcal{O}_L$-modules. Since tensor products commute with colimits we have

$$\varinjlim_n \mathbf{A} \otimes_{\mathcal{O}_L} A_n \cong \mathbf{A} \otimes_{\mathcal{O}_L} A$$

algebraically. But the direct limit topology of $\varinjlim_n \mathbf{A} \otimes_{\mathcal{O}_L} A_n$ again is discrete and so the above isomorphism is also topological. Then, $\mathbf{A} \otimes_{\mathcal{O}_L} A$ is a discrete H_L-module and therefore we deduce from [NSW15, (1.5.1) Proposition, p. 45–46]

$$H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = \varinjlim_n H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_n)$$

for all $i \geq 0$. Since $H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_n) = 0$ for all $i > 0$ and $n \in \mathbb{N}$ we also have $H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ for all $i > 0$. $\square$

Lemma 5.2.8.
Let A be a cofinitely generated $\mathcal{O}_L$-module. Then the sequence

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr} \otimes \mathrm{id} - \mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

is exact and has a continuous set theoretical splitting, where all terms are equipped with the discrete topology.

Proof.
Since $\mathbf{A}$ is a flat $\mathcal{O}_L$-module the first assertion comes from Lemma 5.1.1, the second is obvious since all terms carry the discrete topology. $\square$

Proposition 5.2.9.
Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then

the exact sequence

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

and the canonical homomorphism

$$(\mathbf{A} \otimes_{\mathcal{O}_L} A)^{H_K} \hookrightarrow C^\bullet_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A)$$

induce quasi isomorphisms

$$C^\bullet_{\mathrm{cts}}(H_K, A) \xrightarrow{\simeq} \mathcal{C}^\bullet_{\mathrm{Fr}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) \xleftarrow{\simeq} \mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A)).$$

Proof.
Since Fr commutes with the action from H_K, the exact sequence

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

clearly is an exact sequence of (discrete) H_K-modules. Then Corollary 2.3.4 says that

$$H^i_{\mathrm{cts}}(H_K, A) \cong \mathcal{H}^i_{\mathrm{Fr}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A),$$

which is exactly the first quasi isomorphism. For the second quasi isomorphism it is with Proposition 2.2.24 enough to show

$$H^i_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = \begin{cases} \mathcal{M}_{K|L}(A) & , \text{ if } i = 0 \\ 0 & , \text{ else .} \end{cases}$$

But this is exactly the above Corollary 5.2.7. □

Corollary 5.2.10.
Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then the following sequence is exact

$$0 \longrightarrow H^0_{\mathrm{cts}}(H_K, A) \longrightarrow \mathcal{M}_{K|L}(A) \xrightarrow{\varphi_{K|L}-\mathrm{id}} \mathcal{M}_{K|L}(A) \longrightarrow H^1_{\mathrm{cts}}(H_K, A) \longrightarrow 0.$$

Proof.
This is the long exact cohomology sequence of

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

combined with $H^1_{\text{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ from Corollary 5.2.7. □

In the next step, we want to replace the above exact sequence with a sequence of $\Lambda_K = \mathcal{O}_L[\![\Gamma_K]\!]$-modules. An idea how to do this gives Nekovář in [Nek07, (8.3.3) Corollary, p. 159] but unfortunately the modules we are working with are not ind-admissible, since $\mathbf{A}$ is no direct limit of finitely generated $\mathcal{O}_L[G_K]$-modules. As in the proof of Theorem 5.1.11 we use limits and colimits to reduce to the case of discrete coefficients.

We want to recall the notation from [Nek07, (8.1.1), p. 148; (8.2.1), p. 157] and from the beginning of [Nek07, (8.3) Infinite extensions, p. 158–159].

Definition 5.2.11.
Let G be a profinite group, $U \leq G$ an open subgroup and M a discrete $\mathcal{O}_L[U]$-module. We then define the induced module to be

$$\mathrm{Ind}_U^G(M) \coloneqq \{f \colon G \to X \mid f(ug) = uf(g) \text{ for all } u \in U, g \in G\}.$$

$\mathrm{Ind}_U^G(M)$ carries a G-action by $(g \cdot f)(\sigma) \coloneqq f(\sigma g)$. Furthermore, if M is a discrete $\mathcal{O}_L[G]$-module define

$${}_UM \coloneqq \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G/U], M).$$

${}_UM$ then again carries a G-action by $(\sigma \cdot (f))(x) \coloneqq \sigma(f(\sigma^{-1}(x)))$.
Let now $H \triangleleft G$ be a closed, normal subgroup and $\mathcal{U}(G; H)$ be the open subgroups of G containing H. Then, for $V, U \in \mathcal{U}(G; H)$ with $V \subseteq U$ the canonical map $G/V \twoheadrightarrow G/U$ induces $\mathcal{O}_L$-linear maps ${}_UM \hookrightarrow {}_VM$ under which the system $({}_UM)_{U \in \mathcal{U}(G;H)}$ becomes a filtered directed system. We then set

$$F_{G/H}(M) \coloneqq \varinjlim_{U \in \mathcal{U}(G;H)} {}_UM.$$

Similar as above, $F_{G/H}(M)$ then also carries an action from G. If $H = \{1\}$ we write $\mathcal{U}(G)$ instead of $\mathcal{U}(G; H)$ and $F_G(M)$ instead of $F_{G/\{1\}}(M)$. Furthermore, we set $\mathcal{U}_K \coloneqq \mathcal{U}(G_K; H_K)$ and we write $F_{\Gamma_K}(M)$ instead of F_{G_K/H_K}. This can lead to an abuse of notation, but it will be clear from the context, which construction is chosen.

Remark 5.2.12.
For the above situation, Nekovář proves in [Nek07, (8.1.3), p. 149] *that*

$$\mathrm{Ind}_U^G(M) \longrightarrow {}_UM, \ f \longmapsto [gU \mapsto g(f(g^{-1}))]$$

is a G-equivariant isomorphism.

Remark 5.2.13.
In the above situation, if $f \in F_{G/H}(M)$ then it exists $U \in \mathcal{U}(G;H)$ such that $f \in {}_UM$. If then $V \in \mathcal{U}(G;H)$ with $V \subseteq U$ we also have $f \in {}_VM$ as well as

$$f(gV) = f(gU)$$

for all $g \in G$.

Remark 5.2.14.
Let G be a group and $H \triangleleft G$ a normal subgroup such that G/H is abelian. Then every subgroup $U \leq G$ with $H \subseteq U$ is normal as well.
In particular, if additionally G is profinite and H is closed, then the elements of $\mathcal{U}(G;H)$ are normal, open subgroups of G containing H. This is of great interest for us, since our application of this theory will be $G = G_K$ and $H = H_K$ and $G = \Gamma_K$ and $H = \{1\}$. In both cases, the factor G/H is Γ_K which is abelian.

Proposition 5.2.15.
Let G be a profinite group, $H \triangleleft G$ a closed, normal subgroup, M a discrete $\mathcal{O}_L[G]$-module and let $U \in \mathcal{U}(G;H)$. Then the compact-open topology on ${}_UM$ is discrete and the G-action on ${}_UM$ is again continuous with respect to this topology.
Furthermore, the transition maps ${}_VM \to {}_{V'}M$ for $V, V' \in \mathcal{U}(G;H)$ with $V' \subseteq V$ are injective, the direct limit topology on $F_{G/H}(M)$ is discrete and its G-action is continuous.

Proof.
Since $U \leq G$ is an open subgroup, the set of cosets G/U is finite and therefore $\mathcal{O}_L[G/U]$ is a finitely generated free $\mathcal{O}_L$-module. So in particular, $\mathcal{O}_L[G/U]$ is compact. Then ${}_UM = \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G/U], M)$ is discrete with respect to the compact open topology since M is discrete. To see that the action from G is continuous on ${}_UM$ it is enough to see that for every $f \in {}_UM$ there exists an open subset $V \subseteq G$ under which f is fixed. Note also that G acts by left multiplication on G/U. So, let $f \in {}_UM$ and let $g_1, \ldots, g_n \in G$ be a set of representatives of the cosets of G/U. Since the action of G on M is continuous and M carries the discrete topology, there exist open subsets

$V_1, \ldots, V_n \subseteq G$ such that g_i is fixed by V_i for all $1 \leq i \leq n$. Then f is fixed by $V := \cap_i V_i$.
The statements on $F_{G/H}(M)$ follow immediately by taking the direct limit. So the statement on the transition maps is left. Let $V, V' \in \mathcal{U}(G; H)$ with $V' \subseteq V$. Then the canonical map $G/V' \to G/V$ is surjective. Then $\mathcal{O}_L[G/V'] \to \mathcal{O}_L[G/V]$ is a surjective $\mathcal{O}_L$-linear homomorphism and since $\mathrm{Hom}_{\mathcal{O}_L}(-, M)$ is left exact, the induced homomorphism ${}_{V'}M \to {}_V M$ is injective. □

In the above situation, under the additional assumption that U is normal in G, Nekovář introduces in [Nek07, (8.1.6.3) Conjugation, p. 151] an action from G/U on ${}_U M$ which will be important for us. We recall this action in the following Remark and we prove the statements.

Remark 5.2.16.
Let G be a profinite group, $U \triangleleft G$ be an open, normal subgroup and M a discrete $\mathcal{O}_L[G]$-module. For $g \in G$ and $f \in \mathrm{Ind}_U^G(M)$ we define $\widetilde{\mathrm{Ad}}(g)(f)$ to be

$$(\widetilde{\mathrm{Ad}}(g)(f))(\sigma) := g(f(g^{-1}\sigma)).$$

This is an action from G on $\mathrm{Ind}_U^G(M)$ which is trivial on U, i.e. it induces an action from G/U on $\mathrm{Ind}_U^G(M)$ which we will denote also by $\widetilde{\mathrm{Ad}}$. Since both, $\mathrm{Ind}_U^G(M)$ and G/U carry the discrete topology, this action is continuous.
Furthermore, this action commutes with the standard action from G and under the isomorphism $\mathrm{Ind}_U^G(M) \cong {}_U M$ from Remark 5.2.12 it corresponds to the G/U-action

$$(\widetilde{\mathrm{Ad}}(gU)(f))(\sigma U) = f(\sigma g U)$$

on ${}_U M$. Then clearly the G-action on ${}_U M$ commutes with this action from G/U and the latter is again continuous.

Proof.
Let $f \in \mathrm{Ind}_U^G(M)$ and $\sigma, g, x \in G$. Then

$$\begin{aligned}(x \cdot (\widetilde{\mathrm{Ad}}(gU)(f)))(\sigma) &= (\widetilde{\mathrm{Ad}}(gU)(f))(\sigma x) \\ &= g(f(g^{-1}\sigma x)) \\ &= g((x \cdot f)(g^{-1}\sigma)) \\ &= (\widetilde{\mathrm{Ad}}(gU)(x \cdot f))(\sigma).\end{aligned}$$

Let α denote the isomorphism from Remark 5.2.12, i.e. $\alpha(f)(\sigma U) = \sigma(f(\sigma^{-1}))$. Then

$$\begin{aligned}\alpha((\widetilde{\mathrm{Ad}}(gU)(f)))(\sigma U) &= \sigma((\widetilde{\mathrm{Ad}}(gU)(f))(\sigma^{-1})) \\ &= \sigma g(f(g^{-1}\sigma^{-1})) \\ &= \alpha(f)(\sigma g U) \\ &= (\widetilde{\mathrm{Ad}}(gU)(\alpha(f)))(\sigma U).\end{aligned}$$

□

Lemma 5.2.17.
Let G be a profinite group and $H \triangleleft G$ a closed, normal subgroup, such that G/H is abelian. Then $\widetilde{\mathrm{Ad}}$ induces a continuous action from G/H on $F_{G/H}(M)$.
In particular, with this action $F_{G/H}(M)$ becomes an $\mathcal{O}_L[\![G/H]\!]$-module.

Proof.
The action from G/H on $F_{G/H}(M)$ is given as follows: For $f \in F_{G/H}(M)$ and $U \in \mathcal{U}(G;H)$ such that $f \in {}_U M$ and $g \in G$ we have

$$\widetilde{\mathrm{Ad}}(gH)(f) = \widetilde{\mathrm{Ad}}(gU)(f).$$

This is well defined, since if $V \in \mathcal{U}(G;H)$ such that $V \subseteq U$ then $f \in {}_V M$ and for $\sigma \in G$ we have

$$\widetilde{\mathrm{Ad}}(gU)(f)(\sigma U) = f(\sigma g U) = f(\sigma g V) = \widetilde{\mathrm{Ad}}(gV)(f)(\sigma V).$$

The action is continuous since the above f is fixed under U/H, which is an open subgroup of G/H.
If f is as above, $x \in \mathcal{O}_L[\![G/H]\!]$ and $\mathrm{pr}_U\colon \mathcal{O}_L[\![G/H]\!] \to \mathcal{O}_L[G/U]$ denotes the canonical projection, then we have

$$\widetilde{\mathrm{Ad}}(x)(f) = \widetilde{\mathrm{Ad}}(\mathrm{pr}_U(x))(f).$$

This again is well defined and makes $F_{G/H}(M)$ into an $\mathcal{O}_L[\![G/H]\!]$-module. □

Proposition 5.2.18.
Let G be a profinite group and $H \triangleleft G$ a closed, normal subgroup such that G/H is abelian. Then $F_{G/H}$ is an exact functor, viewed as functor from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[\![G/H]\!][G]$-modules.

Proof.

The above Lemma 5.2.17 says that $F_{G/H}$ is a functor from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[\![G/H]\!][G]$-modules. So it is left to check that it is exact. For fixed $U \in \mathcal{U}(G;H)$ the functor $M \mapsto {}_UM$ from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[G/U][G]$-modules is exact since $\mathcal{O}_L[G/U]$ is a finitely generated, free $\mathcal{O}_L$-module. Since taking direct limits is exact as well, $F_{G/H}$ is exact. □

Definition 5.2.19.
If $\mathbf{C}$ is an abelian category, we denote by $\mathbf{D}(\mathbf{C})$ the corresponding derived category. As usual, we denote by $\mathbf{D}^+(\mathbf{C})$ the full subcategory whose objects are the complexes, which have no nonnegative entries and by $\mathbf{D}^{\mathrm{b}}(\mathbf{C})$ the full subcategory consisting whose objects are the bounded below complexes.
If $\mathfrak{C}^\bullet$ is a complex in an abelian category $\mathbf{C}$, we denote as in [Nek07] by $\mathbf{R}\Gamma(\mathfrak{C}^\bullet)$ the corresponding complex as an object in the derived category $\mathbf{R}(\mathbf{C})$.
In particular, if G is a profinite group and M is a topological G-module we set

$$\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, M) \coloneqq \mathbf{R}\Gamma(C^\bullet_{\mathrm{cts}}(G, M))$$

as an object in $\mathbf{R}(\mathbf{Ab})$.

Remark 5.2.20.
Let G be a profinite group, $H \triangleleft G$ a closed, normal subgroup, and M a discrete $\mathcal{O}_L[G]$-module. As in [Nek07, (3.6.1.4), p. 72] *we define an action from G on $C^\bullet_{\mathrm{cts}}(H, M)$ by*

$$\mathrm{Ad}(g)(c)(h_0, \dots, h_n) \coloneqq g(c(g^{-1}h_0g, \dots, g^{-1}h_ng)),$$

where $c \in C^n_{\mathrm{cts}}(H, M)$. In loc. cit. Nekovář also proves that for $h \in H$ this action is homotopic to the identity and therefore induces an action from G/H on $\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(H, M)$ and $H^*(H, M)$ respectively.
Similarly, by

$$C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \xrightarrow{\widetilde{\mathrm{Ad}}(g)_*} C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \xrightarrow{\mathrm{Ad}(g)} C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$$

we can define an action from G on $C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$. Note that in this situation $\mathrm{Ad}(g)\colon C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \to C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$ is homotopic to the identity and so the complex $\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$ becomes a complex of $\mathcal{O}_L[\![G/H]\!]$-modules. See also Remark 5.2.23 *below.*

Proposition 5.2.21.
Let G be a profinite group, $H \lhd G$ a closed, normal subgroup and M a discrete $\mathcal{O}_L[G]$-module. Then there is a canonical morphism of complexes

$$C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \to C^\bullet_{\mathrm{cts}}(H, M),$$

which is a quasi isomorphism. Moreover, for $g \in G$ the diagram

$$\begin{tikzcd} C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] \arrow[d, "\widetilde{\mathrm{Ad}}(g)"'] & C^\bullet_{\mathrm{cts}}(H, M) \arrow[dd, "\mathrm{Ad}(g)"] \\ C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[d, "\mathrm{Ad}(g)"'] & \\ C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] & C^\bullet_{\mathrm{cts}}(H, M) \end{tikzcd}$$

is commutative. So in particular, the corresponding isomorphism $\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \to \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(H, M)$ in the derived category $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$ is G/H-linear.

Proof.
For the proof set $\mathfrak{U} := \mathfrak{U}(G; H)$. [NSW15, (1.5.1) Proposition, p. 45–46] says that we have

$$C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \cong C^\bullet_{\mathrm{cts}}(G, \varinjlim_{U \in \mathfrak{U}} {}_U M) \cong \varinjlim_{U \in \mathfrak{U}} C^\bullet_{\mathrm{cts}}(G, {}_U M).$$

With Remark 5.2.12 we then obtain

$$\varinjlim_{U \in \mathfrak{U}} C^\bullet_{\mathrm{cts}}(G, {}_U M) \cong \varinjlim_{U \in \mathfrak{U}} C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)).$$

Shapiro's Lemma (cf. [NSW15, (1.6.4) Proposition, p. 62–63]) and again [NSW15, (1.5.1) Proposition, p. 45–46] then give us

$$\varinjlim_{U \in \mathfrak{U}} C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)) \simeq \varinjlim_{U \in \mathfrak{U}} C^\bullet_{\mathrm{cts}}(U, M) \cong C^\bullet_{\mathrm{cts}}(\varprojlim_{U \in \mathfrak{U}} U, M) = C^\bullet_{\mathrm{cts}}(H, M).$$

[Nek07, (8.1.6.3), p. 151] says that for $U \in \mathcal{U}(G; H)$ and $g \in G$ the diagram

$$\begin{tikzcd} C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}_U^G(M)) \arrow[r] \arrow[d, "\widetilde{\mathrm{Ad}}(g)"'] & C^\bullet_{\mathrm{cts}}(U, M) \arrow[dd, "\mathrm{Ad}(g)"] \\ C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}_U^G(M)) \arrow[d, "\mathrm{Ad}(g)"'] & \\ C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}_U^G(M)) \arrow[r] & C^\bullet_{\mathrm{cts}}(U, M) \end{tikzcd}$$

is commutative. Taking direct limits then proves the commutativity of the desired diagram. □

Corollary 5.2.22.
Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{K|L})$ such that M is discrete is $\mathcal{O}_L[G]$-module. Then the above Proposition 5.2.21 together with Proposition 2.2.24 induces the Γ_K-linear isomorphism

$$\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, F_{\Gamma_K}(M))) \xrightarrow{\cong} \mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(M)).$$

Remark 5.2.23.
In the situation of Proposition 5.2.21, the morphism

$$\mathrm{Ad}(g) \colon C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \longrightarrow C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$$

for $g \in G$ is homotopic to the identity (cf. [Nek07, (3.6.1.4), p. 72] *respectively* Remark 5.2.20*) and therefore the diagram*

$$\begin{tikzcd} \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] \arrow[d, "\widetilde{\mathrm{Ad}}(g)_*"'] & \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(H, M) \arrow[d, "\mathrm{Ad}(g)"] \\ \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] & \mathbf{R}\Gamma_{\mathrm{cts}}(H, M) \end{tikzcd}$$

is commutative. The corresponding diagram for cohomology groups

$$\begin{tikzcd} H^*_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] \arrow[d, "\widetilde{\mathrm{Ad}}(g)_*"'] & H^*_{\mathrm{cts}}(H, M) \arrow[d, "\mathrm{Ad}(g)"] \\ H^*_{\mathrm{cts}}(G, F_{G/H}(M)) \arrow[r] & H^*_{\mathrm{cts}}(H, M) \end{tikzcd}$$

then also is commutative. This then explains that the statement from [NSW15, p. 65] *coincides with the theory from Nekovář .*

Proposition 5.2.24.
Let $A = \varinjlim_m A_m$ be a cofinitely generated $\mathcal{O}_L$-module, where $A_m = \ker(\mu_{\pi_L^m})$ as usual, with a continuous action from G_K and set

$$\begin{aligned}
\mathcal{A}_{mn} &\coloneqq (\mathbf{A} \otimes_{\mathcal{O}_L} A_m) \,/\, \big(\pi_L^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} A_m\big) \\
M_{mn} &\coloneqq (\mathbf{A} \otimes_{\mathcal{O}_L} A_m)^{H_K} \,/\, \big(\pi_L^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} A_m\big)^{H_K}.
\end{aligned}$$

Then the following diagram is commutative and each arrow in it is a quasi isomorphism. Moreover, the vertical arrows on the right hand side are homomorphisms of Λ_K-modules.

$$\begin{tikzcd}
C^\bullet_{\mathrm{cts}}(H_K, A) & C^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \arrow[l, "\simeq"'] \\
\varinjlim\limits_{m\in\mathbb{N}} C^\bullet_{\mathrm{cts}}(H_K, A_m) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & \varinjlim\limits_{m\in\mathbb{N}} C^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A_m)) \arrow[l, "\simeq"'] \arrow[u, "\cong"] \arrow[d, "\simeq"'] \\
\varinjlim\limits_{m\in\mathbb{N}} \varprojlim\limits_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\mathrm{Fr}}(H_K, \mathcal{A}_{mn}) & \varinjlim\limits_{m\in\mathbb{N}} \varprojlim\limits_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, F_{\Gamma_K}(\mathcal{A}_{mn})) \arrow[l, "\simeq"'] \\
\varinjlim\limits_{m\in\mathbb{N}} \varprojlim\limits_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(M_{mn}) \arrow[u, "\simeq"'] & \varinjlim\limits_{m\in\mathbb{N}} \varprojlim\limits_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, F_{\Gamma_K}(M_{mn})) \arrow[l, "\simeq"'] \arrow[u, "\simeq"] \\
\varinjlim\limits_{m\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A_m)) \arrow[u, "\cong"'] \arrow[d, "\simeq"] & \\
\mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A)). &
\end{tikzcd}$$

In particular, the induced isomorphism $\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A))) \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))$ in $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$ is Λ_K-linear, i.e. it is an isomorphism in $\mathbf{D}^+(\Lambda_K\text{-}\mathbf{Mod})$.

Proof.

We start with the left column and we consider the following diagram

$$\begin{tikzcd}
C^{\bullet}_{\mathrm{cts}}(H_K, A) & \\
\varinjlim_{m\in\mathbb{N}} C^{\bullet}_{\mathrm{cts}}(H_K, A_m) \arrow[u, "(1)"', "\simeq"] \arrow[d, "(2)"', "\simeq"] & \\
\varinjlim_{m\in\mathbb{N}} \mathcal{C}^{\bullet}_{\mathrm{Fr}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_m) \arrow[dr, "(5)", "\cong"'] & \\
& \varinjlim_{m\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}} \mathcal{C}^{\bullet}_{\mathrm{Fr}}(H_K, \mathcal{A}_{mn}) \\
& \varinjlim_{m\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(M_{mn}) \arrow[u, "\simeq", "(7)"'] \\
\varinjlim_{m\in\mathbb{N}} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A_m)) \arrow[uuu, "(3)"', "\simeq"] \arrow[ur, "\cong", "(6)"'] \arrow[d, "(4)"', "\simeq"] & \\
\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A)). &
\end{tikzcd}$$

That the morphisms (1) and (4) are quasi isomorphisms is well known (cf. eg. [NSW15, (1.5.1) Proposition, p. 45–46]). (2) and (3) are quasi isomorphisms by Proposition 5.2.9. Proposition 2.3.7 says that (5) and (6) are isomorphisms of complexes. But then (7) is also a quasi isomorphism. So, all the morphisms in the left column of the original diagram are at least quasi isomorphisms. The horizontal morphisms are quasi isomorphisms by Proposition 5.2.21 and therefore the morphisms in the right column are also quasi isomorphisms. So it is left to check that the induced isomorphism $\mathbf{R}\Gamma(\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A))) \cong \mathbf{R}\Gamma^{\bullet}_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))$ is Λ_K-linear. But the morphisms

$$\varinjlim_{m\in\mathbb{N}} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A_m)) \longrightarrow \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A))$$

and

$$\varinjlim_{m\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(M_{mn}) \longrightarrow \varinjlim_{m\in\mathbb{N}} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\mathcal{M}_{K|L}(A_m))$$

are clearly Λ_K-linear and so are all the morphisms in the right column of the original diagram with respect to the Λ_K-action induced by $\widetilde{\mathrm{Ad}}$ (which is the correct action in the derived category according to Remark 5.2.23). Finally, the morphism

$$\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}}\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, F_{\Gamma_K}(M_{mn}))) \longrightarrow \varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}}\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(M_{mn}))$$

is Λ_K-linear by Corollary 5.2.22. □

This description now has the advantage that the objects of the complexes are Λ_K-modules which allows us to apply the theory of Matlis duality. We give a brief overview of this theory.

Remark 5.2.25.
We have to consider different types of group actions on Λ_K. First, Γ_K acts by multiplication and G_K acts by multiplication through the natural projection $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$. Sometimes we also have to consider Λ_K as Λ_K-module via the involution ι, i.e. Γ_K then acts by $\gamma\cdot x := \gamma^{-1}x$. If this is the case, we write Λ_K^ι. Note that this does also affect the action from G_K, i.e. G_K acts on Λ_K^ι by $g\cdot x = \mathrm{pr}(g)^{-1}x$ and Γ_K acts by $\gamma\cdot x = \gamma^{-1}x$.
Additionally, if M is a Λ_K-module, we denote by M^ι the Λ_K-module M where Γ_K acts via the involution ι, i.e. for all $\gamma\in\Gamma_K$ and $m\in M$ we have $\gamma\cdot m = \gamma^{-1}m$. If N is another Λ_K-module we clearly have

$$\mathrm{Hom}_{\Lambda_K}(M, N^\iota) = \mathrm{Hom}_{\Lambda_K}(M^\iota, N).$$

Definition 5.2.26.
A Λ_K-module with a Λ_K-semilinear action of G_K is a Λ_K-module M with an action from G_K such that for all $\lambda\in\Lambda_K$, $m\in M$ and $g\in G_K$ we have

$$g(\lambda m) = g(\lambda)g(m) = \mathrm{pr}(g)\lambda g(m),$$

where $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$ denotes the canonical projection (cf. Remark 5.2.25).

Remark 5.2.27.
For us it feels more natural to consider Λ_K-modules with a semilinear from G_K-action instead of Λ_K-modules with a linear action from G_K, which are considered in [Nek07]*. The main reason for this is that if we consider modules with a linear action from G_K we would have to consider Λ_K with the trivial action from G_K. But this feels unintuitive. In the text below we will always compare our results to the results of Nekovář in* [Nek07]*. He considers Λ_K with the trivial action of G_K (cf.* [Nek07, (8.4.3.1) Lemma, p. 161–162]*).*

Both concepts are linked in the following sense: If M is a Λ_K-module with a (linear or semilinear) action from G_K, then for $n \in \mathbb{Z}$ denote by $M < n >$ the Λ_K-module M with the G_K-action given by

$$g \cdot m = \mathrm{pr}(g)^n g(m),$$

whith $g \in G_K$ and $m \in M$ and where $g(m)$ denotes the given action of G_K on M (cf. [Nek07, (8.4.2), p. 161]). Then $M \mapsto M < 1 >$ induces a morphism from Λ_K-modules with a linear action from G_K to Λ_K-modules with a semilinear action from G_K. Its inverse clearly is $M \mapsto M < -1 >$.

Remark 5.2.28.
Let M, N be Λ_K-modules with a Λ_K-semilinear action of G_K. Then $\mathrm{Hom}_{\Lambda_K}(M, N)$ also carries actions from both G_K and Γ_K (respectively Λ_K). The action from Γ_K is given by the multiplication of Λ_K on N (respectively M since the homomorphisms are Λ_K-linear). The action from G_K is given by

$$(g \cdot f)(m) := g_N(f(g_M^{-1}(m))),$$

for $f \in \mathrm{Hom}_{\Lambda_K}(M, N)$ and $m \in M$ and where g_M respectively g_N denote the actions from G_K on M and N.

Remark 5.2.29.
Let T be a topological $\mathcal{O}_L$-module with a continuous action from G_K and let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then Γ_K acts on $\mathrm{Hom}_{\mathcal{O}_L}(T, M)$ by multiplication on the coefficients and G_K as in the above Remark 5.2.28, i.e. by

$$(g \cdot f)(t) := g_M(f(g_T^{-1}(t))),$$

for $f \in \mathrm{Hom}_{\mathcal{O}_L}(T, M)$ and $m \in M$ and where g_T and g_M denote the actions from G_K on T and M respectively.

Lemma 5.2.30.
Let T be a topological $\mathcal{O}_L$-module with a continuous action from G_K and let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then the homomorphism of $\mathcal{O}_L$-modules

$$\mathrm{Hom}_{\mathcal{O}_L}(T, M) \longrightarrow \mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K, M), \; f \longmapsto \beta_f := [t \otimes x \mapsto x f(t)]$$

is an isomorphism which respects the actions from Γ_K and G_K described in the above Remark 5.2.29 for the left hand side and Remark 5.2.25 for the right hand side.

Proof.
The inverse homomorphism is given by

$$\mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K, M) \longrightarrow \mathrm{Hom}_{\mathcal{O}_L}(T, M),\ h \longmapsto [t \mapsto h(t \otimes 1)].$$

So it is left to check that the above homomorphisms respects the actions from Γ_K and G_K. We start with the action from G_K. For this, we have to check, that $g \cdot \beta_f = \beta_{(g \cdot f)}$ holds for all $g \in G_K$ and $f \in \mathrm{Hom}_{\mathcal{O}_L}(T, M)$. So take $f \in \mathrm{Hom}_{\mathcal{O}_L}(T, M)$ and let $t \in T$ and $x \in \Lambda_K$. For $g \in G_K$ we then get

$$\begin{aligned}(g \cdot \beta_f)(t \otimes x) &= g(\beta_f(g^{-1}(t \otimes x))) \\ &= g(\beta_f((g^{-1}(t)) \otimes (g^{-1}(x)))) \\ &= g(g^{-1}(x) f(g^{-1}(t))) \\ &= xgf(g^{-1}(t)) \\ &= x(g \cdot f)(t) \\ &= \beta_{(g \cdot f)}(t \otimes x).\end{aligned}$$

For the fourth line, note that G_K acts semilinear on M. For the action of Γ_K recall, that Γ_K acts on both sides by multiplication on the coefficients. For $\gamma \in \Gamma_K$ we then get

$$\begin{aligned}(\gamma \cdot \beta_f)(t \otimes x) &= \gamma \beta_f(t \otimes x) \\ &= \gamma x f(t) \\ &= x(\gamma f(t)) \\ &= x((\gamma \cdot f)(t)) \\ &= \beta_{(\gamma \cdot f)}(t \otimes x).\end{aligned}$$

□

Remark 5.2.31.
Let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then $M^\vee = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(M, L/\mathcal{O}_L)$ also carries actions from G_K and Γ_K. Both are given by

$$(g \cdot f)(m) = f(g^{-1}(m)),$$

where $g \in G_K$ or in Γ_K, $f \in M^\vee$ and $m \in M$.
Note that Nekovář considers the Pontrjagin dual of M with the Γ_K-action without the involution, i.e. by $(\gamma \cdot f)(m) = f(\gamma(m))$ (cf. the proof respectively the result

of [Nek07, (8.4.3.1) Lemma, p. 161–162]*). In our notation the Pontrjagin dual of Nekovář of M is $(M^\vee)^\iota = (M^\iota)^\vee$.*

Lemma 5.2.32.
Let M be a Λ_K-module with a Λ_K-semilinear action of G_K and $n \in \mathbb{Z}$. Then the identity of $M^\vee$ induces an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K

$$(M < n >)^\vee \cong M^\vee < n > .$$

Proof.
We have to check that the identity of $M^\vee$ is G_K-linear with respect to the above actions. So let $g \in G_K$, $m \in M$ and $f \in M^\vee$ and denote by $\mathrm{pr}\colon G_K \to \Gamma_K$ the canonical projection. For a clearer representation we index g by the module it acts on, e.g. if we consider the action from g on $M < n >$ we write $g_{M<n>}$. On the left hand side we have

$$\begin{aligned}(g_{(M<n>))^\vee} \cdot f)(m) &= f(g^{-1}_{M<n>} \cdot m) \\ &= f(\mathrm{pr}(g)^{-n} g_M^{-1}(m)).\end{aligned}$$

In the first line we used the definition of the action from G_K on the Pontrjagin dual from the above Remark 5.2.31 and in the second line we used the definition of $< n >$. On the right hand side we have

$$\begin{aligned}(g_{M^\vee<n>} \cdot f)(m) &= ((\mathrm{pr}(g)^n) \cdot (g_{M^\vee} \cdot f))(m) \\ &= (\mathrm{pr}(g)^n \cdot f)(g_M^{-1}(m)) \\ &= f(\mathrm{pr}(g)^{-n} g_M^{-1}(m)).\end{aligned}$$

□

Definition 5.2.33.
Let M be a Λ_K-module. The **Matlis dual** of M is defined as

$$\overline{\mathrm{D}_K}(M) := \mathrm{Hom}_{\Lambda_K}(M, \Lambda_K^\vee).$$

This is a contravariant functor of Λ_K-modules and maps finitely generated Λ_K-modules to cofinitely generated and vice versa.
Λ_K acts on $\overline{\mathrm{D}_K}(M)$ by multiplication and if M has also a semilinear action from G_K, then G_K acts on $\overline{\mathrm{D}_K}(M)$ as described in the above Remark 5.2.28

Remark 5.2.34.
$\Lambda_K^\vee$ is an injective Λ_K-module. Moreover, it is an injective hull of the residue class field of Λ_K as Λ_K-module. Therefore $\overline{\mathrm{D}_K}$ is exact and for every finitely respectively cofinitely generated Λ_K-module the canonical homomorphism $M \to \overline{\mathrm{D}_K}(\overline{\mathrm{D}_K}(M))$ is an isomorphism.

Proof.
Since $\gamma \mapsto \gamma^{-1}$ defines an isomorphism of Λ_K-modules $\Lambda_K \to \Lambda_K^\iota$, the first statement is [Nek07, (8.4.3.2) Corollary, p. 162]. For this, note that in [Nek07, (8.4.3.1) Lemma, p. 161–162] Nekovář proves that $(\Lambda_K^\vee)^\iota = (\Lambda_K^\iota)^\vee$ and Nekovář's dualizing module coincide and with $(\Lambda_K^\iota)^\vee$ also $\Lambda_K^\vee$ is a dualizing module. The second statement is [BH98, Theorem 3.2.12, p. 105–107]. □

Remark 5.2.35.
As mentioned in [Nek07, (2.3.3, p. 41)] *$L/\mathcal{O}_L$ is an injective hull for k_L. Therefore we have a canonical isomorphism $M \cong \mathrm{Hom}_{\mathcal{O}_L}(\mathrm{Hom}_{\mathcal{O}_L}(M, L/\mathcal{O}_L), L/\mathcal{O}_L)$ for every finitely or cofinitely generated $\mathcal{O}_L$-module M and $\mathrm{Hom}_{\mathcal{O}_L}(-, L/\mathcal{O}_L)$ is an exact functor. As above, the proof for this is* [BH98, Theorem 3.2.12, p. 105–107].

We need some more notation from [Nek07].

Remark 5.2.36.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and $U \in \mathfrak{U}_K$. Then we have two group actions on $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$. The first action, is the diagonal action from G_K

$$g \cdot (a \otimes xU) = (ga) \otimes (gxU).$$

The second action is the following action from G_K/U:

$$\widetilde{\mathrm{Ad}}(gU)(a \otimes xU) := a \otimes xg^{-1}U.$$

The homomorphism $\sum a_{xU} \otimes xU \mapsto \sum a_{xU}\delta_{xU}$ where δ_{xU} is the Kronecker delta-function on G_K/U (i.e. it is 1 for xU and zero otherwise) defines an isomorphism between $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$ and ${}_UT$ (cf. [Nek07, (8.1.3), p. 149; (8.2.1) p. 157]*) under which the actions described above coincide with the corresponding actions on ${}_UT$ (cf.* [Nek07, (8.1.6.3), p. 151]*).*

Definition 5.2.37.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. We set

$$\mathcal{F}_{\Gamma_K}(T) := \varprojlim_{U \in \mathfrak{U}_K} T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$$

together with the two actions from G_K and Γ_K described in the above Remark 5.2.36. With this, we define

$$\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T) \coloneqq \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, \mathcal{F}_{\Gamma_K}(T)).$$

Furthermore, by $\overset{\mathbf{L}}{\otimes}_R$ we denote the derived tensor product over the ring R.

Remark 5.2.38.
At [Nek07, p. 201] *Nekovář proves*

$$H^*_{\mathrm{Iw}}(K_\infty|K, T) \cong H^*(\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T)),$$

i.e. that the cohomology of the above complex coincides with the Iwasawa cohomology defined in Definition 4.3.6.

Remark 5.2.39.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$, then we have an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K

$$\mathcal{F}_{\Gamma_K}(T) \cong T \otimes_{\mathcal{O}_L} \Lambda_K^\iota.$$

Proof.
Since T is finitely generated and $\mathcal{O}_L$ is a discrete valuation ring, T is finitely presented. Therefore we have

$$\varprojlim_{U \in \mathfrak{U}_K} T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U] = T \otimes_{\mathcal{O}_L} \Lambda_K$$

as $\mathcal{O}_L$-modules. G_K acts on both sides diagonally and Γ_K acts on the left hand side via $\widetilde{\mathrm{Ad}}$ (which technically means via the involution) on the right hand term $\mathcal{O}_L[G_K/U]$. Since Γ_K acts on Λ_K^ι also via the involution, the claim follows. □

Lemma 5.2.40.
We have an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K

$$(\Lambda_K^\iota)^\vee \cong F_{\Gamma_K}(L/\mathcal{O}_L) \left(= \varinjlim_{U \in \mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L) \right).$$

Proof.
$\mathcal{O}_L[G_K/U]$ is compact for $U \in \mathfrak{U}_K$, therefore $\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ is discrete and so $\varinjlim_{U \in \mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ is discrete too. This means that every map with source $\varinjlim_{U \in \mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ into any topological space is

continuous. We then compute (as $\mathcal{O}_L$-modules)

$$\begin{aligned}
\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L) = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(&\varinjlim_{U\in\mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
=&\mathrm{Hom}_{\mathcal{O}_L}(\varinjlim_{U\in\mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
\cong& \varprojlim_{U\in\mathfrak{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
\cong& \varprojlim_{U\in\mathfrak{U}_K} \mathcal{O}_L[G_K/U] \\
=&\Lambda_K.
\end{aligned}$$

At the third equation, we used the identification

$$\mathcal{O}_L[G_K/U] \cong \mathrm{Hom}_{\mathcal{O}_L}(\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L)$$

from Remark 5.2.35. Now we head towards the action from Γ_K. For $\gamma \in \Gamma_K$, $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L)$ and $h \in F_{\Gamma_K}(L/\mathcal{O}_L)$ we have

$$(\gamma \cdot f)(h) = f(\gamma^{-1} \cdot h) = f(\widetilde{\mathrm{Ad}}(\gamma^{-1})h)$$

for all $x \in F_{\Gamma_K}(L/\mathcal{O}_L)$. Going through the above isomorphisms shows that this results in an action from Γ_K on Λ_K via the involution, i.e. we have an isomorphism of Λ_K-modules

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L) \cong \Lambda_K^{\iota}.$$

With the above notation, we have for $g \in G_K$

$$(g \cdot f)(h) = f(g^{-1} \cdot h) = f(h \circ g),$$

since G_K acts trivial on $L/\mathcal{O}_L$ by definition. Therefore the above isomorphism is also G_K-linear. □

Remark 5.2.41.
The above result differs a bit from Nekovář's result in [Nek07, (8.4.3.1) Lemma, p. 161–162] *since Nekovář's considers Λ_K-modules with a Λ_K-linear action from G_K and therefore he considers Λ_K with a trivial G_K action (cf.* Remark 5.2.27*). Furthermore, his Pontrjagin dual and ours for Λ_K-modules differ in the action of Γ_K by an involution (cf.* Remark 5.2.31*). For a better comparison, if we consider Λ_K*

with the trivial action from G_K the result of loc. cit in our notation is

$$(\Lambda_K^\vee)^\iota \cong F_{\Gamma_K}(L/\mathcal{O}_L) < 1 > .$$

This is equivalent to

$$(\Lambda_K^\vee)^\iota < -1 > \cong F_{\Gamma_K}(L/\mathcal{O}_L)$$

and for the left hand side we obtain

$$\begin{aligned}(\Lambda_K^\vee)^\iota < -1 > &= (\Lambda_K^\iota)^\vee < -1 > \\ &= (\Lambda_K^\iota < -1 >)^\vee \\ &= ((\Lambda_K < 1 >)^\iota)^\vee.\end{aligned}$$

In the second line we used Lemma 5.2.32. But this means that Nekovář's result translate into ours since we considered Λ_K with the action from G_K given by the canonical projection $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$.

Lemma 5.2.42.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. Then we haven an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K:

$$F_{\Gamma_K}(T^\vee) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)).$$

Proof.
This proof follows the idea of [Nek07, (8.4.5.1) Lemma, p. 163].
We have

$$\begin{aligned}F_{\Gamma_K}(T^\vee) &= \varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], T^\vee) \\ &= \varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], \mathrm{Hom}_{\mathcal{O}_L}(T, L/\mathcal{O}_L)) \\ &\cong \varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U] \otimes_{\mathcal{O}_L} T, L/\mathcal{O}_L) \\ &\cong \varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(T, \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)).\end{aligned}$$

In the third and fourth line above we used the usual tensor-hom adjunction (cf. Lemma 2.2.29). The above isomorphism is both, G_K- and Γ_K-linear. To see this, it is enough to show that for $U \in \mathcal{U}_K$ the isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], T^\vee) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_L}(T, \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)) \\ f & \longmapsto & \alpha_f := [t \mapsto [x \mapsto f(x)(t)]] \end{array}$$

is G_K and G_K/U-linear. We start with the action from G_K. Let $t \in T$ and $x \in \mathcal{O}_L[G_K/U]$. Let furthermore $f \in \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], T^\vee)$ and $g \in G_K$. The action of G_K on the left hand side is given by

$$(g \cdot f)(x)(t) = g(f(g^{-1}(x))(t)) = f(g^{-1}(x))(g^{-1}(t)).$$

Let $h \in \mathrm{Hom}_{\mathcal{O}_L}(T, \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L))$. Then the action of G_K on the right hand side is given by

$$(g \cdot h)(t)(x) = g(h(g^{-1}(t))(x)) = h(g^{-1}(t))(g^{-1}(x)).$$

Therefore we obtain

$$\begin{aligned} (g \cdot \alpha_f)(t)(x) &= \alpha_f(g^{-1}(t))(g^{-1}(x)) \\ &= f(g^{-1}(x))(g^{-1}(t)) \\ &= (g \cdot f)(x)(t) \\ &= \alpha_{(g \cdot f)}(t)(x). \end{aligned}$$

For $\gamma \in G_K/U$ the action of G_K/U on the left hand side is given by

$$(\gamma \cdot f)(x)(t) = f(\widetilde{\mathrm{Ad}}(\gamma^{-1})(x))(t)$$

and on the right hand side by

$$(\gamma \cdot h)(t)(x) = \gamma(h(t)(x)) = h(t)(\widetilde{\mathrm{Ad}}(\gamma^{-1}(x))).$$

Analogously to the above computation we then obtain

$$\begin{aligned} (\gamma \cdot \alpha_f)(t)(x) &= \alpha_f(t)(\widetilde{\mathrm{Ad}}(\gamma^{-1})(x)) \\ &= f(\widetilde{\mathrm{Ad}}(\gamma^{-1})(x))(t) \\ &= (\gamma \cdot f)(x)(t) \\ &= \alpha_{(\gamma \cdot f)}(t)(x). \end{aligned}$$

Then, as in [Nek07, (8.4.5.1) Lemma, p. 163], since T is finitely generated over $\mathcal{O}_L$,

we have

$$\begin{aligned}&\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(T, \mathrm{Hom}_{\mathcal{O}_L}((\mathcal{O}_L[G_K/U], L/\mathcal{O}_L))) \\ \cong\ & \mathrm{Hom}_{\mathcal{O}_L}(T, \varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)).\end{aligned}$$

With the above Lemma 5.2.40, which says that we have an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K $\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L) \cong (\Lambda_K^\iota)^\vee$, we then deduce

$$\begin{aligned}F_{\Gamma_K}(T^\vee) &\cong \mathrm{Hom}_{\mathcal{O}_L}(T, (\Lambda_K^\iota)^\vee) \\ &\cong \mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K, (\Lambda_K^\iota)^\vee) \\ &\cong \mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K^\iota, \Lambda_K^\vee) \\ &\cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)).\end{aligned}$$

In the second line we used Lemma 5.2.30, in the third line Remark 5.2.25 and in the last Remark 5.2.39. The references for the second and last line also show that the isomorphism is G_K- and Γ_K-linear. For the first line, this is part of this proof and in the third line it is obvious. So the above homomorphism is both G_K- and Γ_K-linear. □

Remark 5.2.43.
Again, the above result differs slightly from the analogous result of Nekovář (cf. [Nek07, (8.4.5.1) Lemma, p. 163]*). This is a consequence of the difference pointed out in the above* Remark 5.2.41*. Translated to our notation, Nekovář's result from loc. cit. then is that there is an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K*

$$F_{\Gamma_K}((T^\vee)^\iota) \cong \mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T)^\iota, (\Lambda_K^\vee)^\iota).$$

Note that Nekovář's original result is formulated for Λ_K-modules with a linear action from G_K. But as pointed out in Remark 5.2.27 *both concepts are linked by the shifts $< 1 >$ and $< -1 >$ respectively. So to be precise, Nekovář's result is the above shifted by $< -1 >$. If we apply this shift, we would have to invert it below in order to compare Nekovář's result to our result. Since Γ_K acts trivially on T and therefore also on $T^\vee$ we have $(T^\vee)^\iota = T^\vee$ and we have a canonical isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K*

$$\mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T)^\iota, (\Lambda_K^\vee)^\iota) = \mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T), \Lambda_K^\vee) = \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)).$$

Combining the above identifications then gives us an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K

$$(F_{\Gamma_K}(T^\vee)) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)),$$

which is exactly our result.

Lemma 5.2.44.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. We then have an isomorphism

$$\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T) \cong \overline{\mathrm{D}_K}\left(\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(T^\vee)(1))\right)[-2].$$

For the cohomology groups we then have for all $i \geq 0$ an isomorphism of Λ_K-modules

$$\overline{\mathrm{D}_K}(H^i_{\mathrm{Iw}}(K_\infty|K, T)) \cong H^{2-i}_{\mathrm{cts}}(G_K, F_{\Gamma_K}(T^\vee(1))) \cong H^{2-i}_{\mathrm{cts}}(H_K, T^\vee(1)).$$

Proof.
This is [Nek07, (8.11.2.2); (8.11.2.3), p. 201], but note that the shift of our complex is outside $\overline{\mathrm{D}_K}(-)$ and that we have $F_{\Gamma_K}(T^\vee) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T))$ (cf. Lemma 5.2.42) since we have a slightly different convention for the involved action of Γ_K. In particular, this is Lemma 5.2.42 together with [Nek07, (5.2.6) Lemma, p. 92]. The last isomorphism of the cohomology groups is Proposition 5.2.21. □

Proposition 5.2.45.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. Then the sequence

$$0 \longrightarrow H^1_{\mathrm{Iw}}(K_\infty|K, T) \longrightarrow \overline{\mathrm{D}_K}(\mathcal{M}) \xrightarrow{\overline{\mathrm{D}_K}(\varphi_{K|L})-\mathrm{id}} \overline{\mathrm{D}_K}(\mathcal{M}) \longrightarrow H^2_{\mathrm{Iw}}(K_\infty|K, T) \longrightarrow 0$$

is exact, where $\mathcal{M} = \mathcal{M}_{K|L}(T^\vee(1))$.

Proof.
With $A := T^\vee(1)$ we deduce from Proposition 5.2.10 and Proposition 5.2.21 that the sequence

$$0 \to H^0_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \to \mathcal{M}_{K|L}(A) \xrightarrow{\varphi_{K|L}-\mathrm{id}} \mathcal{M}_{K|L}(A) \to H^1_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \to 0$$

is exact and Proposition 5.2.24 says that it is a sequence of Λ_K-modules. Applying

$\overline{\mathrm{D}_K}(-)$ then gives the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathrm{D}_K}(H^1_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))) \arrow[r] & \overline{\mathrm{D}_K}(\mathcal{M}_{K|L}(A)) \arrow[r, "\overline{\mathrm{D}_K}(\varphi_{K|L})-\mathrm{id}"] & \cdots \\ \cdots \arrow[r] & \overline{\mathrm{D}_K}(\mathcal{M}_{K|L}(A)) \arrow[r] & \overline{\mathrm{D}_K}(H^0_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))) \arrow[r] & 0 \end{tikzcd}$$

(cf. Remark 5.2.34). Lemma 5.2.44 translates this sequence into the desired one. □

This sequence looks similar to the sequence

$$0 \to H^1_{\mathrm{Iw}}(K_\infty|K, T) \to \mathcal{M}_{K|L}(T(\tau^{-1})) \xrightarrow{\psi-\mathrm{id}} \mathcal{M}_{K|L}(T(\tau^{-1})) \to H^2_{\mathrm{Iw}}(K_\infty|K, T) \to 0$$

from Theorem 4.3.13 where $\tau^{-1} = \chi_{\mathrm{LT}}\chi_{\mathrm{cyc}}^{-1}$ and $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. In order to compare these sequences, we prove the following.

Lemma 5.2.46.
Let $n \in \mathbb{N}$. We have $\Omega^1_{\mathbf{A}_{K|L}}/\pi_L^n\Omega^1_{\mathbf{A}_{K|L}} = (\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L})^\vee$ and a Γ_K-linear inclusion

$$\Omega^1_{\mathbf{A}_{K|L}}/\pi_L^n\Omega^1_{\mathbf{A}_{K|L}} \hookrightarrow \overline{\mathrm{D}_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L})$$

Proof.
The isomorphism is a reformulation of Remark 4.2.16. For the inclusion using the tensor-hom adjunction (cf. Lemma 2.2.29) we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, L/\mathcal{O}_L) &\cong \mathrm{Hom}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L} \otimes_{\Lambda_K} \Lambda_K, L/\mathcal{O}_L) \\ &\cong \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, \mathrm{Hom}_{\mathcal{O}_L}(\Lambda_K, L/\mathcal{O}_L)). \end{aligned}$$

So we have to check that under this isomorphism $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, L/\mathcal{O}_L)$ is sent to $\mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, (\Lambda_K)^\vee)$. For this, recall the above isomorphism precisely: Let $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, L/\mathcal{O}_L)$, then f is mapped to the element

$$[a \mapsto f_a \coloneqq [\lambda \mapsto f(\lambda a)]]$$

in $\mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, \mathrm{Hom}_{\mathcal{O}_L}(\Lambda_K, L/\mathcal{O}_L))$. For $a \in \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}$ the homomorphism f_a then is the composition

$$\begin{tikzcd} \Lambda_K \arrow[r] & \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L} \arrow[r, "f"] & L/\mathcal{O}_L \\ \lambda \arrow[r, mapsto] & \lambda a & \end{tikzcd}$$

of continuous maps, i.e. f_a is continuous too and we get the desired inclusion

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, L/\mathcal{O}_L) \hookrightarrow \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, (\Lambda_K)^\vee).$$

It is left to check this inclusion is Γ_K-linear. For $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, L/\mathcal{O}_L)$ as above, we denote by α_f its image in $\mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}, (\Lambda_K)^\vee)$. Let $\gamma \in \Gamma_K$, $a \in \mathbf{A}_{K|L}/\pi_L^n\mathbf{A}_{K|L}$ and $\lambda \in \Lambda_K$. Then we have

$$\begin{aligned}(\gamma \cdot \alpha_f)(a)(\lambda) &= \gamma(\alpha_f(\gamma^{-1}(a))(\lambda)) \\ &= (f(\gamma^{-1}a\lambda)) \\ &= (\gamma \cdot f)(a\lambda) \\ &= \alpha_{(\gamma\cdot f)}(a)(\lambda).\end{aligned}$$

In the first and the third line we used the definition of the action on the homomorphisms. In the second line we used that Γ_K acts trivially on $L/\mathcal{O}_L$ by definition. $\square$

Definition 5.2.47.
Let M be a toplogical $\mathbf{A}_{K|L}$-module with a continuous and semilinear action from Γ_K. We define

$$\mathcal{D}(M) \coloneqq \mathrm{Hom}_{\mathbf{A}_{K|L}}(M, \Omega^1_{\mathbf{A}_{K|L}} \otimes_{\mathbf{A}_{K|L}} \mathbf{B}_{K|L}/\mathbf{A}_{K|L}).$$

And we define the Γ_K-action on $\mathcal{D}(M)$ to be

$$(\gamma \cdot f)(m) \coloneqq \gamma(f(\gamma^{-1}(m))),$$

where Γ_K acts diagonal on the tensor product.

Remark 5.2.48.
By Proposition 4.2.29 we can identify $\mathcal{D}(M)$, for M as above, with

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M, \mathbf{B}_{K|L}/\mathbf{A}_{K|L}(\chi_{\mathrm{LT}})).$$

Lemma 5.2.49.
Let M be a discrete $\mathbf{A}_{K|L}$-module with a continuous and semilinear action from Γ_K such that $M = \varinjlim_m M_m$ where $M_m = \ker(\mu_{\pi_L^m})$. Then we have a Γ_K-linear inclusion

$$\mathcal{D}(M) \hookrightarrow \overline{\mathrm{D}_K}(M).$$

Proof.

For $m \in \mathbb{N}$ we obtain with the tensor-hom adjunction (cf. Lemma 2.2.29)

$$\begin{aligned}\overline{\mathrm{D}_K}(M_m) &\cong \mathrm{Hom}_{\Lambda_K}(M_m, (\Lambda_K)^\vee) \\ &\cong \mathrm{Hom}_{\Lambda_K}(M_m \otimes_{\mathbf{A}_{K|L}} \mathbf{A}_{K|L}/\pi_L^m \mathbf{A}_{K|L}, (\Lambda_K)^\vee) \\ &\cong \mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^m \mathbf{A}_{K|L}/, (\Lambda_K)^\vee)).\end{aligned}$$

Lemma 5.2.46 then implies, that there is an inclusion

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}}/\pi_L^m \Omega^1_{\mathbf{A}_{K|L}}) \hookrightarrow \overline{\mathrm{D}_K}(M_m).$$

But since $\pi_L^m M_m = 0$ it is

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}}/\pi_L^m \Omega^1_{\mathbf{A}_{K|L}}) = \mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}} \otimes_{\mathbf{A}_{K|L}} \mathbf{B}_{K|L}/\mathbf{A}_{K|L}),$$

i.e. we have an inclusion $\mathcal{D}(M_m) \hookrightarrow \overline{\mathrm{D}_K}(M_m)$. Since $\mathrm{Hom}_R(-, X)$ commutes with limits for arbitrary rings R and R-modules X, we get the desired inclusion $\mathcal{D}(M) \hookrightarrow \overline{\mathrm{D}_K}(M)$ by applying limits. □

Lemma 5.2.50.
Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then we have

$$\mathcal{D}(\mathcal{M}_{K|L}(A)) \cong \mathcal{M}_{K|L}(A^\vee(\chi_{\mathrm{LT}})).$$

This isomorphism respects the action from Γ_K.

Proof.
As usual we write $A = \varinjlim_m A_m$ with $A_m = \ker(\mu_{\pi_L^m})$. Lemma 4.2.17 then says that we have an isomorphism

$$\mathcal{D}(\mathcal{M}_{K|L}(A_m)) \cong \mathcal{M}_{K|L}(A_m)^\vee.$$

Proposition 4.2.35 implies that this isomorphism is Γ_K-linear. Remark 4.3.4 says that we have a Γ_K-linear isomorphism

$$\mathcal{M}_{K|L}(A_m)^\vee \cong \mathcal{M}_{K|L}((A_m)^\vee(\chi_{\mathrm{LT}})).$$

Combining these results gives us the Γ_K-linear isomorphism

$$\mathcal{D}(\mathcal{M}_{K|L}(A_m)) \cong \mathcal{M}_{K|L}((A_m)^\vee(\chi_{\mathrm{LT}})).$$

Applying limits now gives the desired result. □

In the Proposition below, we are using a result of Section 4. Since $K|L$ was unramified in this chapter, we also have from now on to assume that $K|L$ is unramified.

Proposition 5.2.51.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and set

$$\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1}))) \coloneqq \mathcal{C}^\bullet_{\mathcal{D}(\varphi)}(\mathcal{D}(\mathcal{M}_{K|L}(T^\vee(1)))[-1].$$

Then the inclusion of complexes

$$\begin{tikzcd} \mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1}))) \arrow[r, hook] & \mathcal{C}^\bullet_{\overline{\mathrm{D}_K}(\varphi)}(\overline{\mathrm{D}_K}(\mathcal{M}_{K|L}(T^\vee(1))))[-1] \arrow[d, equal] \\ & \overline{\mathrm{D}_K}(\mathcal{C}^\bullet_\varphi(\mathcal{M}_{K|L}(T^\vee(1))))[-2] \end{tikzcd}$$

is a quasi isomorphism. So in particular we have an isomorphism in the derived category $\mathbf{D}^{\mathrm{b}}(\Lambda_K - \mathbf{Mod})$

$$\mathbf{R}\Gamma(\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1})))) \cong \mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K,T).$$

Proof.
With $T^\vee(1) = T(-1)^\vee$, the above Lemma 5.2.49 and Lemma 5.2.50 imply

$$\begin{tikzcd} \mathcal{M}_{K|L}(T(\tau^{-1}) \arrow[r, "\cong"] & \mathcal{D}(\mathcal{M}_{K|L}(T^\vee(1)) \arrow[r, hook] & \overline{\mathrm{D}_K}(\mathcal{M}_{K|L}(T^\vee(1)). \end{tikzcd}$$

The cited lemmata also show that both homomorphisms are Γ_K-linear. Let $\mathcal{M} \coloneqq \mathcal{M}_{K|L}(T^\vee(1))$ then Proposition 5.2.45 together with Theorem 4.3.13 implies the commutative diagram with exact rows and Λ_K-linear vertical homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{Iw}}(K_\infty|K,T) \arrow[r] & \overline{\mathrm{D}_K}(\mathcal{M}) \arrow[r, "\overline{\mathrm{D}_K}(\varphi)-\mathrm{id}"] & \overline{\mathrm{D}_K}(\mathcal{M}) \arrow[r] & H^2_{\mathrm{Iw}}(K_\infty|K,T) \arrow[r] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{Iw}}(K_\infty|K,T) \arrow[r] \arrow[u, equal] & \mathcal{M}_{K|L}(T(\tau^{-1})) \arrow[r, "\psi-\mathrm{id}"] \arrow[u, hook] & \mathcal{M}_{K|L}(T(\tau^{-1})) \arrow[r] \arrow[u, hook] & H^2_{\mathrm{Iw}}(K_\infty|K,T) \arrow[r] \arrow[u, equal] & 0. \end{tikzcd}$$

This gives the desired quasi isomorphism. The second statement then follows from Lemma 5.2.44 by using Proposition 5.2.24. $\square$

Theorem 5.2.52.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and let $K \subseteq K' \subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ is isomorphic to some $\mathbb{Z}_p^r$. Then we have an isomorphism in the derived category $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_{K'}, T).$$

In particular, we have

$$\mathbf{R}\Gamma(\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1})))) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_{K'}, T).$$

Proof.
The first assertion is [Nek07, (8.4.8.1) Proposition, p. 168]. Note that we have an isomorphism $\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K', T) \cong \mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T)$ in $\mathbf{D}^+(\Lambda_{K'}\text{-}\mathbf{Mod})$ since the intermediate fields of $K_\infty|K'$ are cofinal in the intermediate fields of $K_\infty|K$. The second assertion then is an application of Proposition 5.2.51. □

Remark 5.2.53.
It is maybe possible the generalize the above Theorem 5.2.52 to the case of general Γ_K. For classical (φ, Γ)-modules one proves the analogous statement first for procyclic and then for general Γ (cf. [Col04, Proposition 5.3.11, Corollary 5.3.12, Proposition 5.3.13, Proposition 5.3.14, p. 101–103]*).*

The above Theorem 5.2.52 gives the desired comparison of continuous cohomology of a representation T with a complex of $(\varphi_{K|L}, \Gamma_K)$-modules related to the operator ψ. Unfortunately, the above statement is only for the continuous cohomology of a subgroup of G_K. The following Corollary manipulates the given representation to get the continuous cohomology of the whole group G_K. In fact, this is an application of Shapiro's Lemma.

Corollary 5.2.54.
Let $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and let $K \subseteq K' \subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ is isomorphic to some $\mathbb{Z}_p^r$. Then we have an isomorphism in the derived category $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(K_\infty|K, T) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/G_{K'}]).$$

Proof.
Recall from Remark 5.2.36 the object $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/G_{K'}]$ together with the diagonal

action from G_K. Since $\mathcal{O}_L[G_K/G_{K'}]$ is a finite free $\mathcal{O}_L$-module we then get an isomorphism in $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$ (cf. [Nek07, (8.2.2), p. 158] and Remark 2.3.12)

$$\begin{aligned}\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, T\otimes_{\mathcal{O}_L}\mathcal{O}_L[G_K/G_{K'}]) &\cong \varprojlim_{n\in\mathbb{N}} \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, T/\pi_L^n T\otimes_{\mathcal{O}_L}\mathcal{O}_L[G_K/G_{K'}])\\ &\cong \varprojlim_{n\in\mathbb{N}} \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_{K'}, T/\pi_L^n T)\\ &\cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_{K'}, T).\end{aligned}$$

Together with Theorem 5.2.52 this is exactly the claim. □

Remark 5.2.55.

We want to give a more concrete statement of the above Theorem 5.2.52. So let as there $T\in\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$ and $K\subseteq K'\subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ is isomorphic to some $\mathbb{Z}_p^r$. Let furthermore $\gamma_1,\dots,\gamma_r$ be a set of generators of $\Gamma_{K'}$. The **Koszul-complex** *$K_\bullet(\Lambda_{K'})$ of $\Lambda_{K'}$ then is the complex*

$$0\longrightarrow \bigwedge^r\Lambda_{K'}\xrightarrow{\mathrm{d}_r}\bigwedge^{r-1}\Lambda_{K'}\xrightarrow{\mathrm{d}_{r-1}}\cdots\longrightarrow\Lambda_{K'}\xrightarrow{\mathrm{d}_1}\mathcal{O}_L\longrightarrow 0,$$

where $\bigwedge^i\Lambda_{K'}$ denotes the i-th exterior algebra of $\Lambda_{K'}$ and

$$d_i(x_1\wedge\cdots\wedge x_i) = \sum_{j=1}^{i}(-1)^{j+1}\mathrm{pr}(x_j)x_1\wedge\cdots\wedge\widehat{x_j}\wedge\cdots\wedge x_i.$$

Here $\widehat{(-)}$ denotes that this entry is omitted and pr *denotes the projection $\Lambda_{K'}\twoheadrightarrow\Lambda_{K'}/(\gamma_1-1,\dots,\gamma_r-1)\cong\mathcal{O}_L$ (cf. [Sta18,* Section 15.28*]). Under the (uncanonical) isomorphism $\Lambda_{K'}\to\mathcal{O}_L[\![X_1,\dots,X_r]\!]$, $\gamma_i-1\mapsto X_i$ the above projection becomes the projection to degree zero. Then by [Mat87,* Theorem 16.5, p. 128–129*] the Koszul-complex $K_\bullet(\Lambda_{K'})$ of $\Lambda_{K'}$ is a free resolution of $\mathcal{O}_L$ and therefore (cf. [Sta18,* Section 15.57, Definition 15.57.15*]) $\mathbf{R}\Gamma(\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1}))) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}}\mathcal{O}_L$ is represented by the complex*

$$(\mathcal{C}^\bullet_\psi(\mathcal{M}_{K|L}(T(\tau^{-1}))) \otimes_{\Lambda_{K'}} K_\bullet(\Lambda_{K'})$$

which then is isomorphic to the complex

$$\mathrm{Tot}\left(\mathcal{M}_{K|L}(T(\tau^{-1}))\otimes_{\Lambda_{K'}}K_\bullet(\Lambda_{K'})\xrightarrow{(\psi-\mathrm{id})\otimes\mathrm{id}}\mathcal{M}_{K|L}(T(\tau^{-1}))\otimes_{\Lambda_{K'}}K_\bullet(\Lambda_{K'})\right)\cong$$

$$\mathrm{Tot}\left(K_\bullet(\mathcal{M}_{K|L}(T(\tau^{-1})))\xrightarrow{\quad K_\bullet(\psi)-\mathrm{id}\quad}K_\bullet(\mathcal{M}_{K|L}(T(\tau^{-1})))\right).$$

Here $K_\bullet(\mathcal{M}_{K'|L}(T(\tau^{-1})))$ *denotes the Koszul-complex of* $\mathcal{M}_{K'|L}(T(\tau^{-1}))$ *which is defined in an analogous way to the Koszul-complex of* $\Lambda_{K'}$*. This last complex then is the generalization of the* ψ*-Herr complex from the classical theory.*

Chapter 6

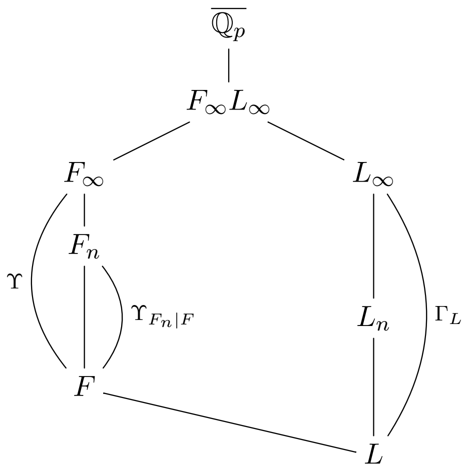
Regulator Maps

In this chapter, we want to define a regulator map, similar to the one in [LZ14a, Definition 4.6, p. 16] and deduce similar properties as in loc. cit. in the following. Besides [LZ14a] (and its previous version [LZ14b]) our main reference for this chapter will be [SV19].

6.1 Notation

We keep the notations from the previous chapters and introduce some new notations, similar to the one of [LZ14a] which will be useful in order to imitate the concepts from there.
First we want to mention that deviant from the previous chapters, we will henceforth denote representations over rings of integers by the letter T while we use V for the corresponding representation over the corresponding quotient field. We do this, because we wanted the notation to be consistent with the other articles in this field.
As in [LZ14a, p. 6] we will work in this chapter mainly with free representations. The category of finitely generated and free $\mathcal{O}_L$-representations of G_K will be denoted by $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg,f})}(G_K)$. Due to the equivalence of Theorem 3.9.1, the $(\varphi_{K|L}, \Gamma_K)$-modules of interest will be those which are finitely generated free as $\mathbf{A}_{K|L}$-modules and we denote the corresponding category by $\mathbf{Mod}_{\varphi,\Gamma}^{\text{ét,f}}(\mathbf{A}_{K|L})$.
Let F be a fixed unramified extension of L, $F_n|F$ be unramified of degree p^n and let $F_\infty = \cup_n F_n$. Note that F_n is uniquely determined. Denote the Galois group of $F_\infty|F$ by Υ and the one of $F_n|F$ by $\Upsilon_{F_n|F}$. Note that since $F_n|F$ is unramified, its Galois group is isomorphic to the Galois group of the extension of the corresponding residue class fields and therefore it is $\Upsilon_{F_n|F} \cong \mathbb{Z}/p^n\mathbb{Z}$. This Galois group is generated by the lift of the q_F-Frobenius from the residue class extension and will be denoted

by σ_{F_n}. It clearly is $\sigma_{F_n}|_{F_{n-1}} = \sigma_{F_{n-1}}$ for all $n \geq 1$. Let Furthermore $\mathcal{O}_{F_n}$ be the ring of integers of F_n and $\mathcal{O}_{F_\infty} = \cup_n \mathcal{O}_{F_n}$.
We want to assemble the whole situation in the following diagram:



Remark 6.1.1.
Don't be confused by the notation, when comparing this section to [LZ14a]*. In* [LZ14a, Proof of Proposition 3.6, p. 10–11] *they define U to be the Galois group of $F_\infty/\mathbb{Q}_p$ and U_n to be the one of $F_\infty|F_n$. But since the groups U_n do not occur in our applications, but the groups U/U_n do, we decided to let our $\Upsilon_{F_n|F}$ be their U/U_n to simplify notation and to be consistent with the definitions we made in previous section (e.g. we defined* $\Gamma_{L_n|L} = \mathrm{Gal}(L_n|L)$ *at the beginning of* Section 3.1*).*

Since we will consider different Iwasawa algebras below, we want to establish the following notation: If G is a profinite group and R a commutative ring, we set

$$\Lambda_R(G) := \varprojlim_{H \triangleleft G} R[G/H],$$

where the projective limit runs over all open normal subgroups of G. If R is a topological ring, we endow $R[G/H]$ with the product topology and $\Lambda_R(G)$ with the topology of the projective limit. Sometimes, this is called the weak topology (cf. [LZ14a, p. 3]).

6.2 Crystalline and Analytic Representations

In this section we want to give a brief overview over some more of Fontaine's period rings and on de Rham and crystalline representations. Since the proofs in our situation are the same as the corresponding ones in [FO10, Chapter 5 and 6, p. 135–198], we will not give a full proof of any of the statements but we will explain how the constructions from loc. cit. transform to our situation. Most of the rings appeared first in [Col02]. But to be consistent with the notation and to simplify comparisons, our main references will be [Sch17] and [FO10].

From [Sch17, Lemma 1.4.18, p. 53–55] we deduce the surjective homomorphism

$$\Theta_{\mathcal{O}_{\mathbb{C}_p^\flat}} : W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \longrightarrow \mathcal{O}_{\mathbb{C}_p^\flat}$$

and in [Sch17, Lemma 2.1.3, p. 86] Schneider proves that its kernel is generated by the element $\xi := \tau(\widetilde{\pi_L}) - \pi_L$, where

$$\tau\colon \mathcal{O}_{\mathbb{C}_p^\flat} \longrightarrow W(\mathcal{O}_{\mathbb{C}_p^\flat})_L,\ x \longmapsto (x, 0, \dots)$$

is the usual Teichmüller Lift (cf. [Sch17, Lemma 1.1.15, p. 15–16]) and $\widetilde{\pi_L} = (\pi_n \bmod \pi_L \mathcal{O}_{\mathbb{C}_p})_n \in \mathcal{O}_{\mathbb{C}_p^\flat}$ with $\pi_0 = \pi_L$ and $\pi_{n+1}^{q_L} = \pi_n$ for all $n \geq 0$ (cf. [Sch17, p. 85]). As in [FO10, p. 92] $\Theta_{\mathcal{O}_{\mathbb{C}_p^\flat}}$ then clearly extends to a surjective homomorphism $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L[1/\pi_L] \to \mathbb{C}_p^\flat$ which we again denote by $\Theta_{\mathcal{O}_{\mathbb{C}_p^\flat}}$. Its kernel then again is generated by ξ. We then define

$$\begin{aligned} \mathbf{B}_{\mathrm{dR}}^+ &\coloneqq \varprojlim_{n\in\mathbb{N}} W(\mathcal{O}_{\mathbb{C}_p^\flat})_L[1/\pi_L]/(\xi)^n, \\ \mathbf{B}_{\mathrm{dR}} &\coloneqq \mathbf{B}_{\mathrm{dR}}^+[1/\xi]. \end{aligned}$$

Remark 6.2.1.
In [Col02, Proposition 7.12, p. 61] *Colmez shows that the above defined ring* $\mathbf{B}_{\mathrm{dR}}^+$ *coincides with the classical de Rham period ring, defined by*

$$\varprojlim_{n\in\mathbb{N}} W(\mathcal{O}_{\mathbb{C}_p^\flat})/(\widetilde{\xi})^n,$$

where $\widetilde{\xi}$ *generates the kernel of the surjective homomorphism* $W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p^\flat}$ *(cf.* [FO10, Definition 5.13, p. 93]*). Note that Colmez denotes the classical ring by* $\mathbf{B}_{\mathrm{dR}}^+$ *and our ring defined above by* $\mathbf{B}_{\mathrm{dR},L}^+$*. Since they coincide, this justifies our notation.*

Remark 6.2.2.
The operation from G_L on $W(\mathcal{O}_{\mathbb{C}_p^\flat})$ carries over to $\mathbf{B}_{\mathrm{dR}}$ and we have (cf. [FO10, Proposition 5.24, p. 96])

$$(\mathbf{B}_{\mathrm{dR}})^{G_L} = L.$$

The role of the element $\pi_\varepsilon = [\varepsilon] - 1$ from [FO10, p. 79] in our situation plays the element ω_ϕ. In particular, it fulfills $\Theta_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\omega_\phi) = 0$ (cf. [Sch17, Lemma 2.1.12, p. 91–92]). Therefore, analogously to [FO10, p. 94], we define

$$t_{\mathrm{LT}} \coloneqq \log_{\mathrm{LT}}(\omega_\phi) \in \mathbf{B}_{\mathrm{dR}}^+$$

As in [FO10, Proposition 5.20, p. 94–95] one then can check that t_{LT} generates the maximal ideal of $\mathbf{B}_{\mathrm{dR}}^+$, using $v(\omega) = \frac{q_L}{q_L - 1}$ from [Sch17, Lemma 1.4.14, p. 50].

Following [FO10, Definition 6.1p. 113–114] we then define

$$\begin{aligned}
\mathbf{A}_{\mathrm{cris}}^0 &\coloneqq \left\{ \sum_{n=0}^{N} a_n \frac{\xi^n}{n!} \,\middle|\, N \in \mathbb{N}_0, a_n \in W(\mathcal{O}_{\mathbb{C}_p^\flat}) \right\}, \\
\mathbf{A}_{\mathrm{cris}} &\coloneqq \varprojlim_{n \in \mathbb{N}} \mathbf{A}_{\mathrm{cris}}^0 / p^n \mathbf{A}_{\mathrm{cris}}^0, \\
\mathbf{B}_{\mathrm{cris}}^+ &\coloneqq \mathbf{A}_{\mathrm{cris}} \left[\frac{1}{p} \right].
\end{aligned}$$

One then can show (cf. [FO10, Proposition 6.6, p. 115]) that there exists another generator t of the maximal ideal of $\mathbf{B}_{\mathrm{dR}}$ such that $t \in \mathbf{A}_{\mathrm{cris}}$. Then we can make the same definition as in [FO10, Definition 6.7, p. 115], i.e. we set

$$\mathbf{B}_{\mathrm{cris}} \coloneqq \mathbf{B}_{\mathrm{cris}}^+ \left[\frac{1}{t} \right] = \mathbf{A}_{\mathrm{cris}} \left[\frac{1}{t} \right].$$

Remark 6.2.3.
As before, the action from G_L restricts to $\mathbf{B}_{\mathrm{cris}}$ and we have

$$(\mathbf{B}_{\mathrm{cris}})^{G_L} = L_0.$$

As usual, we define

$$\begin{aligned}
\mathbf{A}_{\mathrm{cris},L} &\coloneqq \mathbf{A}_{\mathrm{cris}} \otimes_{\mathcal{O}_{L_0}} \mathcal{O}_L, \\
\mathbf{B}_{\mathrm{cris},L}^+ &\coloneqq \mathbf{B}_{\mathrm{cris}}^+ \otimes_{L_0} L, \\
\mathbf{B}_{\mathrm{cris},L} &\coloneqq \mathbf{B}_{\mathrm{cris}} \otimes_{L_0} L.
\end{aligned}$$

One can show $t_{\mathrm{LT}} \in \mathbf{B}_{\mathrm{cris},L}$. For $V \in \mathbf{Rep}_L^{(\mathrm{fg})}(G_L)$ we then also define

$$\begin{aligned}
\mathcal{D}_{\mathrm{dR}}(V) &= \left(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{G_L}\\
\mathcal{D}_{\mathrm{cris}}(V) &= \left(\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{G_L}\\
\mathcal{D}_{\mathrm{cris},L}(V) &= (\mathbf{B}_{\mathrm{cris},L} \otimes_L V)^{G_L} = (\mathbf{B}_{\mathrm{cris}} \otimes_{L_0} V)^{G_L}
\end{aligned}$$

Then $\mathcal{D}_{\mathrm{dR}}(V)$ is an L-vector space and we have (cf. [FO10, p. 98])

$$\dim_L \mathcal{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p}(V).$$

Similarly, $\mathcal{D}_{\mathrm{cris}}(V)$ is an L_0-vector space and it is (cf. [FO10, p. 131])

$$\dim_{L_0} \mathcal{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p}(V).$$

If in the above line holds equality, we say the representation V is **crystalline**. Clearly, $\mathcal{D}_{\mathrm{cris},L}(V)$ is an L-vector space with

$$\dim_L \mathcal{D}_{\mathrm{cris},L}(V) \leq \dim_{L_0} \mathcal{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p}(V).$$

We also want to recall some notation from [SV19]. For this, recall from [FO10, p. 99] that for $V \in \mathbf{Rep}_L^{(\mathrm{fg})}(G_L)$ the K-vector space $\mathcal{D}_{\mathrm{dR}}(V)$ has a filtration, which we will denote by $\mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(V)$ and as in [FO10, p. 100] we set

$$\mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(V) = \mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1}\mathcal{D}_{\mathrm{dR}}(V).$$

The **Hodge-Tate weights** of V then are the $i \in \mathbb{Z}$ with $\mathrm{gr}^i\mathcal{D}_{\mathrm{dR}}(V) \neq 0$. We say that V is **positive** if all the Hodge-Tate weights are ≤ 0. Furthermore, we say that V is **analytic** if the filtration on $\mathcal{D}_{\mathrm{dR}}(V)_{\mathfrak{m}}$ is trivial for every maximal ideal $\mathfrak{m}$ of $L \otimes_{\mathbb{Q}_p} L$ which is not the kernel of the homomorphism $L \otimes_{\mathbb{Q}_p} L \to L$ induced by the multiplication. We denote the full subcategory of $\mathbf{Rep}_L^{(\mathrm{fg})}(G_L)$ of crystalline and analytic representations by $\mathbf{Rep}_L^{\mathrm{cris,an}}(G_L)$.
A free $\mathcal{O}_L$-representation $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg,f})}(G_L)$ is called **crystalline** respectively **analytic** respectively **positiv** if $V := T \otimes_{\mathcal{O}_L} L = T[1/\pi_L]$ is crystalline respectively analytic respectively positive. The corresponding full subcategory of $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg,f})}(G_L)$ of crystalline and analytic $\mathcal{O}_L$-representations is denoted by $\mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$.
Furthermore, set $Q_\phi := \frac{[\pi_L]_\phi \omega_\phi}{\omega_\phi}$ and for a finite extension $\mathcal{K}|L$ we denote by $\mathbf{Mod}_{\varphi,\Gamma}^{\mathrm{an}}(\mathbf{A}_{\mathcal{K}|L}^+)$ the category consisting of finitely free $\mathbf{A}_{\mathcal{K}|L}^+$-modules N, together

with a $\varphi_{\mathcal{K}|L}$-linear homomorphism $\varphi_N \colon N \to N[1/Q_\phi]$ such that

$$1 \otimes \varphi_N \colon \mathbf{A}^+_{\mathcal{K}|L} \, {}_{\varphi_{\mathcal{K}|L}} \otimes_{\mathbf{A}^+_{\mathcal{K}|L}} N[1/Q_\phi] \to N[1/Q_\phi]$$

is an isomorphism and with a semilinear $\Gamma_{\mathcal{K}}$-action, commuting with φ_N such that the induced action on $N/\omega_\phi N$ is trivial. We call such a module an **analytic** $(\varphi_{\mathcal{K}|L}, \Gamma_{\mathcal{K}})$-module over $\mathbf{A}^+_{\mathcal{K}|L}$.

Let $X|\mathbb{Q}_p$ be a finite extension and $E|X$ be an extension, such that E is complete. Let furthermore W be an n-dimensional X-vector space and $B \subseteq W$ be a closed polydisk, i.e. there exists a $w \in W$ and an $s > 0$ such that

$$B = \{b \in W \mid ||b - w|| \leq s\}.$$

For $W = X^n$ we could choose $\mathcal{O}_X^n$ for B, where $\mathcal{O}_X$ is the ring of integers of X (this will be the most interesting of our applications, especially in the case $n = 1$). A function $f \colon B \to E$ is called **locally X-analytic** (with values in E), if for every $b \in B$ there exists an $r > 0$ and a convergent power series $f_b \in X[\![X_1, \ldots, X_n]\!]$ such that

$$f(a + b) = f_b(a)$$

for all $a \in B$ with $||w - b||_W \leq r$. In our applications X will be L or $\mathbb{Q}_p$. Note that a locally $\mathbb{Q}_p$-analytic function needs not to be locally L-analytic. We will denote the E-vector space of all E-valued locally X-analytic functions on B by $C^{X\text{-an}}(B, E)$. We then also get the notion of locally X-analytic functions on Lie groups over X by an analogous definition locally on the charts. If G is a Lie group over X, we denote by $D_X(G, E)$ the continuous dual of the E-vector space $C^{X\text{-an}}(G, E)$, i.e.

$$D_X(G, E) = \mathrm{Hom}_E^{\mathrm{cts}}(C^{X\text{-an}}(G, E), E).$$

$D_X(G, E)$ is called the space of **E-valued locally X-analytic distributions on G**. As in [LZ14a, p. 3] we endow $D_X(G, E)$ with the topology of the inverse limit.
The most interesting case for us will be when G is (a subgroup of) the group of units of the ring of integers $\mathcal{O}_X$ of X.

Remark 6.2.4.
Let $E|\mathbb{Q}_p$ be an extension, such that E is complete. We want to recall from [LZ14a, p. 3] *a way to think of the Iwasawa algebra and the distributions in one of the cases we are interested in. Let $G \cong \Delta \times \mathbb{Z}_p^n$, where Δ is a finite abelian group. Let $\gamma_1, \ldots, \gamma_n$ denote a set of generators of the $\mathbb{Z}_p^n$ factor. Then $\Lambda_{\mathcal{O}_E}(G)$ is isomorphic to $\mathcal{O}_E[\Delta][\![X_1, \ldots, X_n]\!]$ via sending $\gamma_i - 1$ to X_i. The space $D_{\mathbb{Q}_p}(G, E)$ then identifies*

with the subring of $E[\Delta][\![X_1, \dots, X_n]\!]$ *consisting of those power series converging on the disk* $|X_i| < 1$.

6.3 On Integral Normal Bases

In their course of proving [LZ14a, Theorem 4.7, p. 16–17], they need a special description for integral normal bases and of the normal bases of the corresponding residue class fields. Since these results are split over three different sources, we collect them here and add some details. First, we fix some notation and then fill in some details in the original proof of the important input, which is [Wae91, p. 203–204].

Definition 6.3.1.
Let $E|F$ be an extension of degree t of finite fields of characteristic p, let q be the cardinality of F, let $x \in E$ and denote by σ the q-Frobenius on both, E and F. The polynomials $f \in F[X]$ with $f(\sigma)(x) = 0$ are called the **annihilating polynomials of x with respect to σ.**
Furthermore, we call an element $x \in E$ a **normal basis generator** of $E|F$ if $(x, \sigma(x), \dots, \sigma^{t-1}(x))$ is a F-basis of E.

Remark 6.3.2.
Let $E|F$ be an extension of degree t of finite fields of characteristic p, let q be the cardinality of F, let $x \in E$ and denote by σ the q-Frobenius on both E and F. Then the annihilating polynomials of x with respect to σ clearly form an ideal and therefore, since $F[X]$ is a principal ideal domain, it exists a unique monic generator of this ideal. We call this generator the **minimal polynomial of x with respect to σ** *and denote it by f_x. Note that these minimal polynomials need not to be irreducible.*

Now we want to fill in a detail into [Wae91, p. 203–204], precisely we want to give a proof of the next-to-last sentence in the proof. The idea of this is to imitate to prove at [Wae91, p. 126] as it is stated on top of [Wae91, p. 204].

Lemma 6.3.3.
Let $E|F$ be an extension of degree t of finite fields of characteristic p, let q be the cardinality of F, let $x, y \in E$ and denote by σ the q-Frobenius on both E and F. If f_x and f_y are relatively prime, then we have $f_x f_y = f_{x+y}$. In particular, if $f \in F[X]$ such that $f(\sigma)(x+y) = 0$ then $f(\sigma)(x) = f(\sigma)(y) = 0$.

Proof.
Since $(f_x f_y)(\sigma)$ is additive we have

$$(f_x f_y)(\sigma)(x + y) = (f_x f_y)(\sigma)(x) + (f_x f_y)(\sigma)(y) = 0$$

and therefore $f_{x+y} \mid f_x f_y$ by definition. Let $g \in F[X]$ such that $f_{x+y}g = f_x f_y$. Now let $P \in F[X]$ be a prime divisor of f_x and let $e \in \mathbb{N}$ be the biggest exponent of P such that $P^e \mid f_x$. Since f_x and f_y are relatively prime, we have $P \nmid f_y$. Then clearly

$$(f_x/P^n)(\sigma)(x) \neq 0$$

for all $1 \leq n \leq e$ and with $P \nmid f_y$ we obtain

$$((f_x f_y)/P^n)(\sigma)(x+y) = ((f_x f_y)/P^n)(\sigma)(x) \neq 0$$

for all $1 \leq n \leq e$. We show now, that we then already have $P^e \mid f_{x+y}$. Let $f \in \mathbb{N}_0$ with $f \leq e$ which is the biggest exponent of P such that $P^f \mid f_{x+y}$. This means that $P^{e-f} \mid g$ and therefore

$$0 = (f_{x+y}g/P^{e-f})(\sigma)(x+y) = ((f_x f_y)/P^{e-f})(\sigma)(x+y) = ((f_x f_y)/P^{e-f})(\sigma)(x).$$

So we get $(f_x/p^{e-f})(\sigma)(x) = 0$ and therefore $e = f$. Imitating this for all prime divisors of f_x and f_y then implies $f_x \mid f_{x+y}$ and $f_y \mid f_{x+y}$ and since f_x and f_y are relatively prime also $f_x f_y \mid f_{x+y}$. □

The statement of [Wae91, p. 203–204] is summarized at [Sem89, Lemma 1, p. 507], which we want to recall here.

Lemma 6.3.4.
Let $E|F$ be an extension of degree t of finite fields of characteristic p, let q be the cardinality of F denote by σ the q-Frobenius on both E and F. Then an element $x \in E$ is a normal basis generator if and only if the minimal polynomial of x with respect to σ is $X^t - 1$.

[Sem89, Lemma 4.1, p. 518] is also one input, we need. We state it here and explain the details.

Lemma 6.3.5.
Let $E|F$ be an extension of degree $t = p^r$ of finite fields of characteristic p, let q be the cardinality of F and denote by σ the q-Frobenius on both E and F. Then an element $x \in E$ is a normal basis generator if and only if $\mathrm{Tr}_{E|F}(x) \neq 0$.
If t is not a power of p and $x \in E$ is a normal basis generator, then we still have $\mathrm{Tr}_{E|F}(x) \neq 0$.

Proof.
Let $x \in E$ be a normal basis generator and t not necessarily a power of p. Since

$\mathrm{Tr}_{E|F}$ is a polynomial of degree $t-1$ in σ it immediately follows from Lemma 6.3.4 that $\mathrm{Tr}_{E|F}(x) \neq 0$ (since its minimal polynomial with respect to σ has degree t). For the other direction, let $t = p^r$ and $x \in E$ such that $\mathrm{Tr}_{E|F}(x) \neq 0$. We then have

$$X^{p^r} - 1 = (X-1)^{p^r} = \left(\sum_{i=0}^{p^r-1} X^i\right)(X-1).$$

Since $\sigma^{p^r} = \mathrm{id}_E$ and $\mathrm{Tr}_{E|F}(x) \neq 0$ and since $X-1$ is the only prime divisor of $X^{p^r}-1$, it immediately follows that $X^{p^r}-1$ is the minimal polynomial of x with respect to σ. □

Then, [Pic18, Theorem 4.12, p. 22] proves in some cases that there are also integral normal bases. Unfortunately, we have from now on to assume $p \neq 2$.

Theorem 6.3.6.
Let $M|L$ be a finite, unramified extension with Galois group G and $[M:L] = p^r$ for some $r \in \mathbb{N}$. Let $x \in \mathcal{O}_M$ such that $\mathrm{Tr}_{\mathcal{O}_M|\mathcal{O}_L}(x) \not\equiv 0 \bmod \pi_L$, then x is an integral normal basis generator, i.e. $(g(x) \mid g \in G)$ is an $\mathcal{O}_L$-basis of $\mathcal{O}_M$ and $x \bmod \pi_L$ is a normal basis generator of the extension $k_M|k_L$.

Proof. This is [Pic18, Theorem 4.12, p. 22]. □

6.4 Yager Modules

In this section, we follow the idea of [LZ14a, Section 3.2, p. 10–11]. Precisely, we follow the earlier version [LZ14b, Section 3.2, p. 7–11] which contains more details and which imitates the construction of [Yag82, §2]. We add some details and explain how this fits in our situation.

Definition 6.4.1.
As in [LZ14b, p. 7], on the ring $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ we define the following group actions from $\Upsilon_{F_n|F}$:

$$\begin{aligned}
\Delta_1\colon \Upsilon_{F_n|F} \times \mathcal{O}_{F_n}[\Upsilon_{F_n|F}] &\longrightarrow \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]\\
\Big(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\Big) &\longmapsto \sum_{g\in\Upsilon_{F_n|F}} h(x_g)\cdot g,\\
\Delta_2\colon \Upsilon_{F_n|F} \times \mathcal{O}_{F_n}[\Upsilon_{F_n|F}] &\longrightarrow \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]\\
\Big(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\Big) &\longmapsto \sum_{g\in\Upsilon_{F_n|F}} x_g\cdot (hg) = \sum_{g\in\Upsilon_{F_n|F}} x_{h^{-1}g}\cdot g.
\end{aligned}$$

The following Remark lists the properties of the above group actions, which are analogous to the ones listed in [LZ14b, p. 7].

Remark 6.4.2.
For every $h \in \Upsilon_{F_n|F}$ the induced map $\Delta_1(h,-)$ on $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is an automorphism of rings, but in general it is not $\mathcal{O}_{F_n}$-linear (though it is $\mathcal{O}_L$-linear). The induced map $\Delta_2(h,-)$ is $\mathcal{O}_{F_n}$-linear but it is in general no homomorphism of rings.
Furthermore, Δ_1 and Δ_2 commute with each other, in the sense that for every $h,k \in \Upsilon_{F_n|F}$ and $r \in \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ we have

$$\Delta_1(k, \Delta_2(h,r)) = \Delta_2(h, \Delta_1(k,r)).$$

Proof.
Let $h \in \Upsilon_{F_n|F}$. That both, $\Delta_1(h,-)$ and $\Delta_2(h,-)$, are additive is clear since addition in $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is just adding the coefficients. For the multiplicativity of $\Delta_1(h,-)$ let $\sum x_g \cdot g$ and $\sum y_g \cdot g$ be in $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ and compute

$$\begin{aligned}
\Delta_1 &\left(h, \left(\sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\right) \cdot \left(\sum_{g\in\Upsilon_{F_n|F}} y_g \cdot g\right)\right) \\
&= \Delta_1\left(h, \sum_{g\in\Upsilon_{F_n|F}} \left(\sum_{ab=g} x_a y_b\right) \cdot g\right) \\
&= \sum_{g\in\Upsilon_{F_n|F}} h\left(\sum_{ab=g} x_a y_b\right) \cdot g \\
&= \sum_{g\in\Upsilon_{F_n|F}} \left(\sum_{ab=g} h(x_a)h(y_b)\right) \cdot g \\
&= \left(\sum_{g\in\Upsilon_{F_n|F}} h(x_g) \cdot g\right)\left(\sum_{g\in\Upsilon_{F_n|F}} h(y_g) \cdot g\right) \\
&= \Delta_1\left(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\right) \Delta_1\left(h, \sum_{g\in\Upsilon_{F_n|F}} y_g \cdot g\right).
\end{aligned}$$

To show that $\Delta_1(h,-)$ is an automorphism of $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ it then remains to show that $\Delta_1(h,-)$ is bijective. But this is clear, since h is an automorphism of $\mathcal{O}_{F_n}$ and $\Delta_1(h,-)$ only acts on the coefficients. It is not $\mathcal{O}_{F_n}$-linear, because if $x \in \mathcal{O}_{F_n} \setminus \mathcal{O}_{F_{n-1}}$

then we have $h(x) \neq x$ if $h \neq \mathrm{id}$ and therefore

$$\Delta_1(h, x \cdot \mathrm{id}) = h(x) \cdot \mathrm{id} \neq x \cdot \mathrm{id} = x(\Delta_1(h, 1 \cdot \mathrm{id})).$$

It clearly is $\mathcal{O}_L$-linear, since the restriction of $\Upsilon_{F_n|F}$ to $\mathcal{O}_L$ is, by definition, trivial. For the $\mathcal{O}_{F_n}$-linearity of $\Delta_2(h, -)$ let $\sum x_g \cdot g \in \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ and $y \in \mathcal{O}_{F_n}$ and compute

$$\begin{aligned}
\Delta_2\left(h, y \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g\right) &= \Delta_2\left(h, \sum_{g \in \Upsilon_{F_n|F}} yx_g \cdot g\right) \\
&= \sum_{g \in \Upsilon_{F_n|F}} yx_g \cdot (hg) \\
&= y \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot (hg) \\
&= y\Delta_2\left(h, \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g\right).
\end{aligned}$$

To see that $\Delta_2(h, -)$ is not multiplicative in general, take $a, b \in \Upsilon_{F_n|F}$ and compute

$$\Delta_2(h, 1 \cdot ab) = 1 \cdot hab$$

as well as

$$\Delta_2(h, 1 \cdot a)\Delta_2(h, 1 \cdot b) = h^2 ab$$

which are equal if and only if $h = \mathrm{id}$.
To see that Δ_1 and Δ_2 commute, take $h, k \in \Upsilon_{F_n|F}$ and $\sum x_g \cdot g \in \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ and compute

$$\begin{aligned}
\Delta_1\left(k, \Delta_2\left(h, \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g\right)\right) &= \Delta_1\left(k, \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot (hg)\right) \\
&= \sum_{g \in \Upsilon_{F_n|F}} k(x_g) \cdot (hg) \\
&= \Delta_2\left(h, \sum_{g \in \Upsilon_{F_n|F}} k(x_g) \cdot g\right) \\
&= \Delta_2\left(h, \Delta_1\left(k, \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g\right)\right).
\end{aligned}$$

□

Next, we want to define a map analogous to the map $y_{K|F}$ from [LZ14a, Definition 3.4, p.10] respectively y from [LZ14b, Definition 3.2, p. 7].

Definition 6.4.3.
We define the following map

$$\begin{aligned} \mu_n \colon \mathcal{O}_{F_n} &\longrightarrow \mathcal{O}_{F_n}[\Upsilon_{F_n|F}] \\ x &\longmapsto \sum_{g \in \Upsilon_{F_n|F}} g^{-1}(x) \cdot g. \end{aligned}$$

Lemma 6.4.4.
The map μ_n is additive, injective and $\mathcal{O}_F$-linear but it is not $\mathcal{O}_{F_n}$-linear in general.

Proof.
That μ_n is additive is clear since $g \in \Upsilon_{F_n|F}$ is an automorphism on $\mathcal{O}_{F_n}$ and the addition on $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is defined by adding the coefficients. It also clearly is injective, since $\mu_n(x) = 0$ implies $g(x) = 0$ for all $g \in \Upsilon_{F_n|F}$ which is only true for $x = 0$.
For $y \in \mathcal{O}_F$ it is $g(y) = y$ for all $g \in \Upsilon_{F_n|F}$ by definition. Therefore we clearly have

$$g(yx) = yg(x)$$

for all $x \in \mathcal{O}_{F_n}$, i.e. μ_n is $\mathcal{O}_L$-linear.
To see that μ_n is not $\mathcal{O}_{F_n}$-linear in general, let $y \in \mathcal{O}_{F_n} \setminus \mathcal{O}_{F_{n-1}}$ and recall from the Proof of the above Remark 6.4.2 $g(y) \neq y$ for $g \neq \mathrm{id}$. For $0 \neq x \in \mathcal{O}_{F_n}$ we then obtain

$$g(yx) = g(y)g(x) \neq yg(x)$$

which immediately implies $\mu_n(yx) \neq y\mu_n(x)$.

□

Following [LZ14b, p. 7] we define the $\mathcal{O}_L$-submodule of $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ in which the actions from Definition 6.4.1 coincide. At [LZ14a, p. 10] is also a description of this, though it is less formal.

Definition 6.4.5.
We define

$$S_n := \left(\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]\right)^{\Delta_1 = \Delta_2}.$$

and equip S_n with the subspace topology of $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$, where $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ itself carries the product topology of the π_L-adic topology on $\mathcal{O}_{F_n}$.

Remark 6.4.6.
Because of Remark 6.4.2 the automorphisms $\Delta_1(h,-)$ for $h \in \Upsilon_{F_n|F}$ of $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ then restrict to automorphisms of S_n. Note that these automorphisms are topological. With Corollary 6.4.8 this is immediately clear.
The automorphism $\Delta_1(\sigma_{F_n},-)$ will be called the Frobenius of S_n.

Remark 6.4.7.
For every $n \in \mathbb{N}$, the π_L-adic topology on $\mathcal{O}_L[\Upsilon_n]$ coincides with the product topology of the π_L-adic topology on $\mathcal{O}_L$.
The analogous statement holds true for $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$.

Proof.
With $n_g \in \mathbb{N}_0$ for $g \in \Upsilon$ we have

$$\pi_L^{\min\{n_g\}}\mathcal{O}_L[\Upsilon_n] \subseteq \prod_{g\in\Upsilon_{F_n|F}} \pi_L^{n_g}\mathcal{O}_L \cdot g \subseteq \pi_L^{\max\{n_g\}}\mathcal{O}_L[\Upsilon_n],$$

which proves that the two topologies coincide.
The proof for $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is the same. □

Corollary 6.4.8.
For every $n \in \mathbb{N}$, the topology on S_n is the π_L-adic topology.

Proof.
We will show that for every $m \in \mathbb{N}_0$ we have

$$\pi_L^m S_n = (\pi_L^m \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]) \cap S_n.$$

The inclusion $\pi_L^m S_n \subseteq (\pi_L^m \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]) \cap S_n$ is obvious. Now let $h \in \Upsilon_{F_n|F}$. Then Remark 6.4.2 says that $\Delta_1(h,-)$ is $\mathcal{O}_L$-linear and $\Delta_2(h,-)$ is even $\mathcal{O}_{F_n}$-linear. Let $x \in \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ such that $\pi_L^m x \in (\pi_L^m \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]) \cap S_n$. Then we have

$$\pi_L^m \Delta_1(h,x) = \Delta_1(h,\pi_L^m x) = \Delta_2(h,\pi_L^m x) = \pi_L^m \Delta_2(h,x).$$

Since $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is a torsion free $\mathcal{O}_L$-module we then deduce

$$\Delta_1(h,x) = \Delta_2(h,x).$$

So we have $x \in S_n$ and $\pi_L^m x \in \pi_L^m S_n$. The claim then follows immediately from the above Remark 6.4.7. □

Lemma 6.4.9.
Multiplication within $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ induces a map $\mathcal{O}_F[\Upsilon_n] \times S_n \to S_n$ by which S_n becomes an $\mathcal{O}_F[\Upsilon_n]$-module. Moreover, the additive map μ_n from Definition 6.4.3 *induces an isomorphism of the $\mathcal{O}_F[\Upsilon_n]$-modules $\mathcal{O}_{F_n}$ and S_n, which respects the Frobenii on both sides, i.e. we have*

$$\mu_n(\sigma_{F_n}(x)) = \Delta_1(\sigma_{F_n}, \mu_n(x))$$

for all $x \in \mathcal{O}_{F_n}$.

Proof.
With Lemma 6.4.4 it remains to show, that μ_n is $\mathcal{O}_F[\Upsilon_n]$-linear and its image is S_n. We start with computing the image of μ_n.
Let $x \in \mathcal{O}_{F_n}$ and $h \in \Upsilon_{F_n|F}$. Then

$$\begin{aligned}
\Delta_1(h, \mu_n(x)) &= \Delta_1\left(h, \sum_{g \in \Upsilon_{F_n|F}} g^{-1}(x) \cdot g\right) \\
&= \sum_{g \in \Upsilon_{F_n|F}} hg^{-1}(x) \cdot g \\
&= \sum_{g \in \Upsilon_{F_n|F}} (h^{-1}g)^{-1}(x) \cdot (hh^{-1}g) \\
&= \sum_{g \in \Upsilon_{F_n|F}} g^{-1}(x) \cdot (hg) \\
&= \Delta_2\left(h, \sum_{g \in \Upsilon_{F_n|F}} g^{-1}(x) \cdot g\right) \\
&= \Delta_2(h, \mu_n(x)).
\end{aligned}$$

So, the image of μ_n is contained in S_n. For the other inclusion let $\sum x_g \cdot g \in S_n$ and $h \in \Upsilon_{F_n|F}$. Since the actions Δ_1 and Δ_2 coincide on S_n by definition, we obtain that

$$\sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g = \sum_{g \in \Upsilon_{F_n|F}} x_g \cdot (hh^{-1}g) \overset{\Delta_1 = \Delta_2}{=} \sum_{g \in \Upsilon_{F_n|F}} h(x_g) \cdot (h^{-1}g) = \sum_{g \in \Upsilon_{F_n|F}} h(x_{hg}) \cdot g,$$

i.e. that $h(x_{hg}) = x_g$ for all $g \in \Upsilon_{F_n|F}$ which is equivalent to $x_{hg} = h^{-1}(x_g)$ for all $g \in \Upsilon_{F_n|F}$ and implies in particular $x_h = h^{-1}(x_1)$. Therefore we get

$$\sum_{g \in \Upsilon_{F_n|F}} x_g \cdot g = \sum_{g \in \Upsilon_{F_n|F}} g^{-1}(x_1) \cdot g = \mu_n(x_1).$$

Since μ_n by Lemma 6.4.4 is additive and $\mathcal{O}_F$-linear it remains to show that it also is $\Upsilon_{F_n|F}$-linear. For $x \in \mathcal{O}_{F_n}$ and $h \in \Upsilon_{F_n|F}$, we compute similar as before

$$\mu_n(h(x)) = \sum_{g\in\Upsilon_{F_n|F}} g^{-1}(h(x)) \cdot g = \sum_{g\in\Upsilon_{F_n|F}} hg^{-1}(x) \cdot g = \Delta_1(h, \mu_n(x)),$$

where we use at the second equality that $\Upsilon_{F_n|F}$ is abelian. The statement on the Frobenii then is the computation above with $h = \sigma_{F_n}$. □

Lemma 6.4.10.
$\mathcal{O}_{F_n}$ is for each $n \in \mathbb{N}$ a free rank 1 module over $\mathcal{O}_F[\Upsilon_{F_n|F}]$.

Proof.
As mentioned in the proof of [LZ14b, Proposition 3.5, p. 8], this is a consequence of Theorem 6.3.6:
Since $F_n|F$ is unramified of degree p^n, Theorem 6.3.6 says that there is an element $x_n \in \mathcal{O}_{F_n}$ which is a normal basis generator of $F_n|F$, i.e. $(g(x_n) \mid g \in \Upsilon_{F_n|F})$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_{F_n}$. Then every element of $\mathcal{O}_{F_n}$ can be written in the form $\sum_{g\in\Upsilon_{F_n|F}} a_g g(x_n)$, with $a_g \in \mathcal{O}_F$, i.e. the $\mathcal{O}_F[\Upsilon_{F_n|F}]$-linear map.

$$\mathcal{O}_F[\Upsilon_{F_n|F}] \longrightarrow \mathcal{O}_{F_n}, \quad \sum_{g\in\Upsilon_{F_n|F}} a_g \cdot g \longmapsto \sum_{g\in\Upsilon_{F_n|F}} a_g g(x_n)$$

is bijective. □

Corollary 6.4.11.
For every $n \in \mathbb{N}$, the π_L-adic topology on $\mathcal{O}_{F_n}$ and its topology as $\mathcal{O}_F[\Upsilon_{F_n|F}]$-module coincide.

Proof.
This now is an immediate consequence of Lemma 6.4.10 and Remark 6.4.7. □

Corollary 6.4.12.
For every $n \in \mathbb{N}$, the isomorphism of $\mathcal{O}_L[\Upsilon_{F_n|F}]$-modules $\mu_n \colon \mathcal{O}_{F_n} \to S_n$ is topological and the canonical inclusion $S_n \hookrightarrow \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ has closed image.

Proof.
Corollary 6.4.8 and Corollary 6.4.11 together say that the isomorphism between $\mathcal{O}_{F_n}$ and S_n induced from μ_n is topological. Since $\mathcal{O}_{F_n}$ is compact with respect to the π_L-adic topology, S_n is compact as well and so is its image in $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$, which then also is closed, since $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ is a Hausdorff space. □

Remark 6.4.13.
Note that the above Corollary 6.4.12 says that S_n is compact, since $\mathcal{O}_{F_n}$ is compact and μ_n is a topological isomorphism.

The next step is to see that the S_n give rise to an inverse system. Due to the above Lemma 6.4.9 this is equivalent to that the $\mathcal{O}_{F_n}$ form an inverse system with respect to the trace maps. For this, we prove [LZ14b, Proposition 3.3, p. 7–8] in our case. Denote by Ξ_n the Galois group of $F_n|F_{n-1}$ and let Tr_n denote the trace map from $\mathcal{O}_{F_n}$ to $\mathcal{O}_{F_{n-1}}$, i.e.

$$\mathrm{Tr}_n\colon \mathcal{O}_{F_n} \longrightarrow \mathcal{O}_{F_{n-1}},\ x \longmapsto \sum_{g\in\Xi_n} g(x).$$

Note that Tr_n induces a homomorphism $\mathcal{O}_{F_n}[\Upsilon_{F_{n-1}|F}] \to \mathcal{O}_{F_{n-1}}[\Upsilon_{F_{n-1}|F}]$ by applying to the coefficients. Furthermore, recall that the canonical projection $\mathrm{pr}_n\colon \Upsilon_{F_n|F} \twoheadrightarrow \Upsilon_{n-1}$ induces a homomorphism of rings

$$\mathcal{O}_{F_n}[\Upsilon_{F_n|F}] \longrightarrow \mathcal{O}_{F_n}[\Upsilon_{F_{n-1}|F}],\ \sum_{g\in\Upsilon_{F_n|F}} x_g\cdot g \longmapsto \sum_{h\in\Upsilon_{n-1}}\left(\sum_{\substack{g\in\Upsilon_{F_n|F}\\ g\equiv h}} x_g\right)\cdot h,$$

which we also will denote by pr_n.

Remark 6.4.14.
The trace map Tr_n commutes with the corresponding Frobenii, i.e. for every $n\in\mathbb{N}$ we have

$$\mathrm{Tr}_n\circ\sigma_{F_n} = \sigma_{F_{n-1}}\circ\mathrm{Tr}_n.$$

Proof.
This follows immediately from $\sigma_{F_n}|_{F_{n-1}} = \sigma_{F_{n-1}}$. □

Proposition 6.4.15.
$(\mathcal{O}_{F_n},\mathrm{Tr}_n)_n$ is an inverse system of $\mathcal{O}_L$-modules with surjective transition maps. Moreover, the composition of homomorphisms

$$\mathcal{O}_{F_n} \xrightarrow{\mu_n} \mathcal{O}_{F_n}[\Upsilon_{F_n|F}] \xrightarrow{\mathrm{pr}_n} \mathcal{O}_{F_n}[\Upsilon_{F_{n-1}|F}]$$

has image in $\mathcal{O}_{F_{n-1}}[\Upsilon_{F_{n-1}|F}]$, i.e. pr_n induces a homomorphism $S_n\to S_{n-1}$ and the *diagram*

$$\begin{tikzcd} \mathcal{O}_{F_n} \arrow[r, "\mu_n"] \arrow[d, "\mathrm{Tr}_n"'] & S_n \arrow[d, "\mathrm{pr}_n"] \\ \mathcal{O}_{F_{n-1}} \arrow[r, "\mu_{n-1}"] & S_{n-1} \end{tikzcd}$$

is commutative. So in particular, $(S_n, \mathrm{pr}_n)_n$ is an inverse system of $\mathcal{O}_L$-modules with surjective transition maps.

Proof.
For the first assertion, the only thing to prove isthe statement that the trace maps are all surjective. The idea for this is at [LZ14b, Proposition 3.5, p. 8]. Since $F_n|F_{n-1}$ is, by definition, unramified, the corresponding extension of the residue class fields $k_{\mathcal{O}_{F_n}}|k_{\mathcal{O}_{F_{n-1}}}$ is separable. Therefore, the trace map between the residue class fields is not zero (cf. [Sta18, Tag 0BIE, Lemma 9.20.7]) and since it is $k_{\mathcal{O}_{F_{n-1}}}$-linear it clearly is surjective. Since $\mathcal{O}_{F_n}$ and $\mathcal{O}_{F_{n-1}}$ are complete with respect to the π_L-adic topology, this then induces that Tr_n also is surjective.
For the second assertion let $x \in \mathcal{O}_{F_n}$ and compute

$$\mathrm{pr}_n(\mu_n(x)) = \mathrm{pr}_n\left(\sum_{g\in\Upsilon_{F_n|F}} g^{-1}(x)\cdot g\right) = \sum_{h\in\Upsilon_{n-1}}\left(\sum_{\substack{g\in\Upsilon_{F_n|F}\\ g\equiv h}} g^{-1}(x)\right)\cdot h$$

For every $h \in \Upsilon_{n-1}$ fix now a lift $\widetilde{h} \in \Upsilon_{F_n|F}$. In particular, if $g \in \Upsilon_{F_n|F}$, such that $g \bmod \Xi_n = h$, then we can find $r \in \Xi_n$ such that $g = r\widetilde{h}$. Then we can rewrite the above equation to

$$\mathrm{pr}_n(\mu_n(x)) = \sum_{h\in\Upsilon_{n-1}}\left(\sum_{r\in\Xi_n} \widetilde{h}^{-1}r^{-1}(x)\right)\cdot h = \sum_{h\in\Upsilon_{n-1}} h^{-1}\left(\sum_{r\in\Xi_n} r(x)\right)\cdot h.$$

Because of

$$\mathrm{Tr}_n(x) = \sum_{r\in\Xi_n} r(x)$$

and since Υ_{n-1} preserves $\mathcal{O}_{F_{n-1}}$ we then get $\mathrm{pr}_n(\mu_n(x)) \in \mathcal{O}_{F_{n-1}}[\Upsilon_{F_{n-1}|F}]$ as desired. Furthermore, we observe

$$\mathrm{pr}_n(\mu_n(x)) = \sum_{h\in\Upsilon_{n-1}} h^{-1}\mathrm{Tr}_n(x)\cdot h = \mu_{n-1}(\mathrm{Tr}_n(x))$$

and since S_{n-1} is the image of μ_{n-1} and Tr_n is surjective, their composition $\mathcal{O}_{F_n} \to$

S_{n-1} is also surjective. So, $\mathrm{pr}_n \colon S_n \to S_{n-1}$ also has to be surjective. □

Definition 6.4.16.
As in [LZ14b, Definition 3.4, p. 8], we define the **Yager module** S_∞ to be

$$S_\infty := \varprojlim_n S_n.$$

Remark 6.4.17.
Since for every $n \in \mathbb{N}$ the Frobenius $\Delta_1(\sigma_{F_n}, -)$ of S_n is a topological automorphism and since these Frobenii commute with the transition maps of the inverse system $(S_n)_n$ (cf. Remark 6.4.14) the projective limit $\varprojlim_n \Delta_1(\sigma_{F_n}, -)$ again is a topological automorphism of S_∞. We will denote this automorphism by φ_{S_∞} and its inverse by ψ_{S_∞}.

The following Lemma is named as a well known fact in [LZ14b, Proposition 3.5, p. 8], but there is no reference. It is also mentioned in [LZ14a, Remark 3.3, p. 10] and [LZ14a, Proposition 3.2, p. 9–10] also fits in our situation. Nevertheless, we will explain the proof using the theory of integral normal bases.

Proposition 6.4.18.
$\varprojlim_n \mathcal{O}_{F_n}$ (and then also S_∞) is a free rank 1-module over $\Lambda_{\mathcal{O}_F}(\Upsilon)$.

Proof.
The idea is to construct a trace compatible system of elements $x_n \in \mathcal{O}_{F_n}$ with $x_0 \not\equiv 0 \bmod \pi_L \mathcal{O}_F$. Because then it is $\mathrm{Tr}_{\mathcal{O}_{F_n}|\mathcal{O}_F}(x_n) \not\equiv 0 \bmod \pi_L \mathcal{O}_F$ and x_n is an integral normal basis generator of $\mathcal{O}_{F_n}|\mathcal{O}_F$ (cf. Theorem 6.3.6) and therefore generates $\mathcal{O}_{F_n}$ as $\mathcal{O}_F[\Upsilon_n]$-module (cf. Lemma 6.4.10). Then $(x_n)_n$ generates $\varprojlim_n \mathcal{O}_{F_n}$ as $\Lambda_{\mathcal{O}_F}(\Upsilon)$-module and is in particular free of rank 1. The existence of such a system is ensured by the surjectivity of the involved trace maps (cf. Proposition 6.4.15):
We start with an element $x_0 \in \mathcal{O}_F$ such that $x_0 \neq 0 \bmod \pi_L \mathcal{O}_F$. Assuming we have $x_n \in \mathcal{O}_{F_n}$ such that $\mathrm{Tr}_{\mathcal{O}_{F_n}|\mathcal{O}_F}(x_n) \not\equiv 0 \bmod \pi_L \mathcal{O}_F$ we choose an element $x_{n+1} \in \mathcal{O}_{F_{n+1}}$ such that $\mathrm{Tr}_{\mathcal{O}_{F_{n+1}}|\mathcal{O}_{F_n}}(x_{n+1}) = x_n$. Then it is

$$\mathrm{Tr}_{\mathcal{O}_{F_{n+1}}|\mathcal{O}_L}(x_{n+1}) = \mathrm{Tr}_{\mathcal{O}_{F_n}|\mathcal{O}_F}(x_n) \not\equiv 0 \bmod \pi_L \mathcal{O}_F.$$

□

As in [LZ14b, Proposition 3.6, p. 8] S_∞ can be realized as a submodule of $\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon) = \varprojlim \mathcal{O}_{F_\infty}[\Upsilon_{F_n|F}]$. We will explain this below.

Remark 6.4.19.
The canonical inclusion $S_\infty \hookrightarrow \Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$ *is a topological embedding with closed image.*

Proof.
Since all the S_n are compact (cf. Remark 6.4.13), their projective limit S_∞ is compact as well. Furthermore, $\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$ is a Hausdorff space since all the $\mathcal{O}_{F_\infty}[\Upsilon_{F_n|F}]$ are Hausdorff spaces and therefore the embedding $S_\infty \hookrightarrow \Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$ is topological with closed image. $\square$

Remark 6.4.20.
As in Definition 6.4.1 we can define two actions from Υ *on* $\widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]$ *and then also on* $\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$*, which we again denote by* Δ_1 *and* Δ_2 *respectively.*

For every $n \in \mathbb{N}$ denote by Θ_n the Galois group of $F_\infty|F_n$.

Lemma 6.4.21.
For every $n \in \mathbb{N}$ *it is*

$$\widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2} = \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2}.$$

Proof.
This is an outline of the last sentence of [LZ14b, Proposition 3.6, p. 8].
For the inclusion $\mathcal{O}_{F_n}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2} \subseteq \widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2}$ is nothing to prove. For the other inclusion, first note, that $\Upsilon_{F_n|F} = \Upsilon/\Theta_n$ and therefore multiplication with elements from Θ_n is trivial on $\Upsilon_{F_n|F}$. This then means, that Θ_n acts trivial on $\widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]$ through Δ_2. In particular, for $h \in \Theta_n$ and $\sum x_g \cdot g \in \widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]$ we have

$$\Delta_2\left(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\right) = \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot (hg) = \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g.$$

If now $\sum x_g \cdot g \in \widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2}$, we can compute

$$\begin{aligned}\sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g &= \Delta_2\left(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\right)\\ &= \Delta_1\left(h, \sum_{g\in\Upsilon_{F_n|F}} x_g \cdot g\right)\\ &= \sum_{g\in\Upsilon_{F_n|F}} h(x_g) \cdot g.\end{aligned}$$

So we get $h(x_g) = x_g$ for all $g \in \Upsilon_{F_n|F}$ and $h \in \Theta_n$, i.e. $x_g \in \left(\widehat{\mathcal{O}}_{F_\infty}\right)^{\Theta_n}$ for all $g \in \Upsilon_{F_n|F}$ and because of $\left(\widehat{\mathcal{O}}_{F_\infty}\right)^{\Theta_n} = \mathcal{O}_{F_n}$ (cf. Lemma 3.2.11) we have $x_g \in \mathcal{O}_{F_n}$ for all $g \in \Upsilon_{F_n|F}$. So we can conclude $\sum x_g \cdot g \in \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]$ and therefore $\widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2} = \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2}$. □

Proposition 6.4.22.
The canonical inclusion $S_\infty \hookrightarrow \Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$ *induces a topological isomorphism* $S_\infty \cong (\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon))^{\Delta_1=\Delta_2}$.

Proof.
Since the canonical inclusion $S_\infty \hookrightarrow \Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon)$ is a topological embedding (cf. Remark 6.4.20), it only is to check that its image is $(\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon))^{\Delta_1=\Delta_2}$. As stated in [LZ14b, Proposition 3.6, p. 8], to see this it is enough to show $\widehat{\mathcal{O}}_{F_\infty}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2} = \mathcal{O}_{F_n}[\Upsilon_{F_n|F}]^{\Delta_1=\Delta_2}$ what is Lemma 6.4.21. □

The following remark is from [LZ14b, p. 8].

Remark 6.4.23.
Let $E|L$ be a finite extension and $\tau\colon \Upsilon \to E^\times$ a continuous character, i.e. a continuous group homomorphism. Then τ induces a homomorphism

$$\Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon) \to \widehat{\mathcal{O}}_{F_\infty} \otimes_{\mathcal{O}_L} \mathcal{O}_E$$

which we also will denote by τ.

Following [LZ14b, p. 8], we can make the same observation.

Proposition 6.4.24.
Let $E|L$ be a finite extension, $\tau\colon \Upsilon \to E^\times$ be a continuous character and $\Omega \in S_\infty$. For $\sigma \in \Upsilon$ we then have

$$\sigma(\tau(\Omega)) = \tau(\sigma)\tau(\Omega),$$

i.e. $\tau(\Omega)$ is a period for the character τ^{-1}.
If Ω is a generator of S_∞ as $\Lambda_{\mathcal{O}_L}(\Upsilon)$-module, then we have $\eta(\Omega) \not\equiv 0 \bmod \pi_L$ for all continuous characters $\eta\colon \Upsilon \to E^\times$.

Proof.
This is exactly [LZ14b, Proposition 3.7, p. 8–9]. □

6.5 Wach Modules

As in [LZ14b, Section 3.3, Section 3.4, p. 9–11] we want to make use of the theory of Wach modules. The references for our situation are [KR09, Corollary 3.3.8, p. 460] and [SV19, p. 6–19].

The following proposition and definition are straight generalizations of [SV19, p. 6–7].

Proposition 6.5.1.
Let $E|L$ be a finite extension. For $V \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_E)$ exists a module $\mathcal{N}_{E|L}(T) \in \mathbf{Mod}^{\mathrm{an}}_{\varphi,\Gamma}(\mathbf{A}^+_{E|L})$ such that the $(\varphi_{E|L}, \Gamma_E)$-structure of $\mathcal{N}_{E|L}(V)$ is induced from the one of $\mathcal{M}_{E|L}(V)$ and we have

$$\mathbf{A}_{E|L} \otimes_{\mathbf{A}^+_{E|L}} \mathcal{N}_{E|L}(V) = \mathcal{M}_{E|L}(V).$$

Definition 6.5.2.
Let $E|L$ be a finite extension. For $V \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_E)$ the module $\mathcal{N}_{E|L}(V) \in \mathbf{Mod}^{\mathrm{an}}_{\varphi,\Gamma}(\mathbf{A}^+_{E|L})$ from the above Proposition 6.5.1 is called the **Wach module** of V over E.

Remark 6.5.3.
As in [SV19, Proposition 1.8, p. 8–10] *one can prove that $\mathcal{N}_{E|L}(V)$ for $V \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_E)$ is unique inside $\mathcal{M}_{E|L}(V)$ with the properties described in* Proposition 6.5.1.

Remark 6.5.4.
We want to topologize the Wach modules in the same way as in [LZ14b, p. 9]. *Since we discussed the weak topology in detail, this fits into a greater picture:*
Let $E|L$ be a finite extension and $V \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_E)$. As all other modules on the $(\varphi_{E|L}, \Gamma_E)$-side, we want to equip $\mathcal{N}_{E|L}(V)$ with a weak topology. To do this consistently, we define it to be the induced topology from $\mathcal{M}_{E|L}(V)$, where $\mathcal{M}_{E|L}(V)$ carries its weak topology. Since $\mathcal{N}_{E|L}(V)$ is a finitely free $\mathbf{A}^+_{E|L}$-module and the weak topology on $\mathbf{A}^+_{E|L}$ is the topology defined by the ideals $(\pi_L, \omega_\phi)^n \mathbf{A}^+_{E|L}$ for $n \geq 0$, a basis for the weak topology on $\mathcal{N}_{E|L}(V)$ is given by the $\mathbf{A}^+_{E|L}$-submodules $(\pi_L, \omega_\phi)^n \mathcal{N}_{E|L}(V)$ for $n \geq 0$. This then coincides with the topology of loc. cit.

Remark 6.5.5.
Recall $Q_\phi = \frac{[\pi_L]_\phi \omega_\phi}{\omega_\phi}$ from the beginning of this chapter. By definition, we then have $\varphi_L(\omega_\phi) = Q_\phi \omega_\phi$.
If $E|L$ is a finite extension, $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_E)$ and $V = T[1/\pi_L]$ recall also that we denoted by $\varphi^* \mathcal{N}_{E|L}(V)$ the $\mathbf{A}^+_{E|L}$-submodule of $\mathcal{N}_{E|L}(V)[1/Q_\phi]$ generated by $\mathrm{im}(\varphi_{\mathcal{N}_{E|L}(V)})$. Using the projection formula for $\psi_{\mathcal{M}_{E|L}(V)}$ (cf. Remark 4.2.3*) and*

the fact that $\mathbf{A}_{E|L}^{+}$ *is stable under* $\psi_{E|L}$*, gives a* ψ*-operator*

$$\psi_{\mathcal{N}_{E|L}(V)}\colon \varphi^*\mathcal{N}_{E|L}(V) \to \mathcal{N}_{E|L}(V).$$

If all Hodge-Tate weights of V *are* ≥ 0*, we have* $\mathcal{N}_{E|L}(V) \subseteq \varphi^*\mathcal{N}_{E|L}(V)$*, i.e.* $\psi_{\mathcal{N}_{E|L}(V)}$ *restricts to an endomorphism of* $\mathcal{N}_{E|L}(V)$ *and we obtain a homomorphism*

$$\mathcal{N}_{E|L}(T)^{\psi=1} \to (\varphi^*\mathcal{N}_{E|L}(V))^{\psi=0},\ x \mapsto x - \frac{\pi_L}{q_L}\varphi_{\mathcal{N}_{E|L}(V)}(x).$$

Lemma 6.5.6.
Let $E|L$ *be a finite and unramified extension and let* $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg},\mathrm{f})}(G_E)$*. Then there exists a canonical isomorphism of* $(\varphi_{E|L}, \Gamma_E)$*-modules*

$$\mathcal{M}_{E|L}(V) \cong \mathcal{M}_L(V) \otimes_{\mathcal{O}_L} \mathcal{O}_E,$$

where φ *on the right hand side is* $\varphi_{\mathcal{M}_L(V)} \otimes \sigma_{E|L}$ *with* $\sigma_{E|L}$ *the Frobenius of the unramified extension* $E|L$*, i.e. the element of the Galois group which raises an element to its* q_L*-th power modulo* π_L*.*

Proof.
The proof is the same as in [LZ14b, Lemma 2.4, p. 4–5]. □

Lemma 6.5.7.
Let $E|L$ *be a finite and unramified extension and let* $V \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris},\mathrm{an}}(G_L)$*. Then there exists a canonical isomorphism*

$$\mathcal{N}_{E|L}(V) \cong \mathcal{N}_L(V) \otimes_{\mathcal{O}_L} \mathcal{O}_E.$$

Proof.
With Lemma 6.5.6 this is an immediate consequence of the uniqueness property of $\mathcal{N}_{E|L}(V)$ inside $\mathcal{M}_{E|L}(V)$ (cf. Remark 6.5.3). □

As in [LZ14b, p. 9] we are interested in the Wach modules over $\mathcal{O}_{F_n}$. The above Lemma 6.5.7 says that they have a special structure coming from the Wach module over L. If $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris},\mathrm{an}}(G_L)$ it then is clear from Proposition 6.4.15 that the $\mathcal{N}_{F_n|L}(T)$ form an inverse system with surjective transition maps, which then leads to the following definition (cf. [LZ14a, Definition 3.10, p. 11]).

Definition 6.5.8.

$$\mathcal{N}_{F_\infty|L}(T) := \varprojlim_{n\in\mathbb{N}} \mathcal{N}_{F_n|L}(T).$$

Remark 6.5.9.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$. Then Lemma 6.5.7 says that we have

$$\mathcal{N}_{F_n|L}(T) \cong \mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} \mathcal{O}_{F_n}.$$

With Lemma 6.4.9 this then transforms into

$$\mathcal{N}_{F_n|L}(T) \cong \mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_n.$$

We now would like to have a similar description for $\mathcal{N}_{F_\infty|L}(T)$. Let for this $\mathfrak{a}$ be the ideal of $\mathbf{A}_{F|L}^+ \otimes_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ generated by $(\pi_L, \omega_\phi, \upsilon - 1)$, where υ is a topological generator of Υ. We denote the completions of $\mathbf{A}_{F|L}^+ \otimes_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ and $\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty$ with respect to the $\mathfrak{a}$-adic topology by $\mathbf{A}_{F|L}^+ \widehat{\otimes}_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ and $\mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty$ respectively. The idea for this construction is taken from [LZ14b, p. 9]*, which unfortunately is no longer a part of the newer version* [LZ14a].
Note that both $\mathbf{A}_{F|L}^+ \widehat{\otimes}_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ and $\mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty$ are compact since the quotients $\mathbf{A}_{F|L}^+ \widehat{\otimes}_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)/(\pi_L, \omega_\phi, \upsilon - 1)^n$ are finite for all $n \in \mathbb{N}$.
Note also, that since $\mathbf{A}_{F|L}^+ \otimes_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ and $\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty$ are Hausdorff spaces, the canonical homomorphisms $\mathbf{A}_{F|L}^+ \otimes_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon) \to \mathbf{A}_{F|L}^+ \widehat{\otimes}_{\mathcal{O}_F} \Lambda_{\mathcal{O}_F}(\Upsilon)$ and $\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty \to \mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty$ are injective (cf. [Bou89a, Chapter III, §3.4, Theorem 1, p.248]*).*

Proposition 6.5.10.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$. We then have an isomorphism

$$\mathcal{N}_{F_\infty|L}(T) \cong \mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty.$$

Proof.
The idea is the same as in [LZ14b, Proposition 3.12, p. 9–10] which did not make its way to [LZ14a, Proposition 3.11, p. 11–12]. Therefore, we recall it here for our situation.
Fix $n \geq 0$. Since the natural projection $S_\infty \to S_n$ is surjective (cf. Proposition 6.4.15) it clearly induces a surjection

$$\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty \twoheadrightarrow \mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_n \cong \mathcal{N}_{F_n|L}(T),$$

which commutes with the transition maps from the inverse system, since the projection $S_\infty \twoheadrightarrow S_n$ does. The kernel of the canonical projection $S_\infty \twoheadrightarrow S_n$ is the $\Lambda_{\mathcal{O}_L}(\Upsilon)$-submodule $\left(\upsilon^{p^n} - 1\right) S_\infty$ and since S_n is a free and therefore flat $\mathcal{O}_L$-module, the kernel of the above homomorphism is the $\mathbf{A}_{F|L}^{+} \otimes_{\mathcal{O}_F} \Lambda_{\mathcal{O}_L}(\Upsilon)$-submodule

$$\left(\upsilon^{p^n} - 1\right) \mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty.$$

Since $\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty$ is a Hausdorff space and $\mathcal{N}_{F_n|L}(T)$ is complete (with respect to their respective weak topologies), the above homomorphism induces a continuous homomorphism

$$\mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty \to \mathcal{N}_{F_n|L}(T)$$

(cf. [Bou89a, Chapter III, §3.4, Proposition 8, p. 248]). It clearly is also surjective, since $\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty \twoheadrightarrow \mathcal{N}_{F_n|L}(T)$ is and the diagram

$$\begin{tikzcd}
\mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty \arrow[rr, hook] \arrow[dr, two heads] & & \mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty \arrow[dl] \\
& S_n &
\end{tikzcd}$$

is commutative. This homomorphism then again commutes with the transition maps of the inverse system $(\mathcal{N}_{F_n|L}(T))_n$ and therefore induces the homomorphism

$$\mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty \to \mathcal{N}_{F_\infty|L}(T).$$

Since the involved modules are compact (cf. Remark 6.5.9 for $\mathcal{N}_{F|L}(T) \widehat{\otimes}_{\mathcal{O}_F} S_\infty$ and $\mathcal{N}_{F_\infty|L}(T)$ is compact since all the $\mathcal{N}_{F_n|L}(T)$ are compact) this is still surjective with kernel

$$\bigcap_{n \in \mathbb{N}} \left(\upsilon^{p^n} - 1\right) \mathcal{N}_{F|L}(T) \otimes_{\mathcal{O}_F} S_\infty = \{0\}.$$

Since it's bijective and continuous and the involved spaces are compact Hausdorff spaces, it is a topological isomorphism. □

6.6 The Regulator Map

The aim of this section is to define a regulator map similar to [LZ14a, Definition 4.6, p.16]. Unfortunately we cannot adopt their whole constructing since in our situation we have no result similar to [LZ14a, Proposition 3.12, p.12] because in the general Lubin-Tate case it is not known if there exists an $\mathbf{A}_L$-basis $(u_1, \ldots, u_{q_L})$ of $\varphi_L(\mathbf{A}_L)$ such that $\psi_L(u_i) = \delta_{1i}$. Therefore, we make a similar construction to [SV19, p. 71] using the ring $\mathcal{R}^+$, which are the power series over a complete extension of L, converging on the open unit disk (for a precise description see [SV19, Section 2.2.1, p. 36–40]). This ring then has the above described property and in Lemma 6.6.4 we prove the statement which in our case plays the part of [LZ14a, Proposition 3.12, p.12].

Lemma 6.6.1.
Let $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_L)$ such that T has no quotient isomorphic to the trivial representation and such that all Hodge-Tate weights of $V := T[1/\pi_L]$ are ≥ 0. We then have

$$H^1_{\mathrm{Iw}}(F_\infty L_\infty|L, T) \cong \mathcal{N}_{F_\infty|L}(T(\tau^{-1}))^{\psi=1}.$$

Proof.
From Theorem 4.3.13 we deduce

$$H^1_{\mathrm{Iw}}(F_n L_\infty|F_n, T) \cong \mathcal{M}_{F_n|L}(T(\tau^{-1}))^{\psi=1}$$

for every $n \in \mathbb{N}$. Since the intermediate fields of $F_n L_\infty|F_n$ are clearly a subset of the intermediate fields of $F_n L_\infty|L$, they are also cofinal and therefore it is

$$H^1_{\mathrm{Iw}}(F_n L_\infty|F_n, T) = H^1_{\mathrm{Iw}}(F_n L_\infty|L, T).$$

As in [SV19, Lemma1.30, p. 21–22] (here we need the assumption that T has no quotient isomorphic to the trivial representation) one then shows

$$\mathcal{M}_{F_n|L}(T(\tau^{-1}))^{\psi=1} = \mathcal{N}_{F_n|L}(T(\tau^{-1}))^{\psi=1}.$$

Taking projective limits then gives us

$$H^1_{\mathrm{Iw}}(F_\infty L_\infty|L, T) \cong \mathcal{N}_{F_\infty|L}(T(\tau^{-1}))^{\psi=1}.$$

□

For the construction of the regulator map we need some more notation and observations from [SV19] respectively from [Col16]. In particular, we recall from [SV19, Section 2.2.1, p. 36–40] the notion of the Robba ring and some properties of it. For this overview let $\mathcal{K}$ be a complete extension of L. Then we denote by $\mathcal{R}^+_{\mathcal{K}}$ the subring of $\mathcal{K}[\![Z]\!]$ consisting of those the power series converging for all $z \in \mathbb{C}_p$ with absolute value less than 1. For on Interval $I \subseteq [0,1]$, we denote by $\mathcal{R}^I_{\mathcal{K}}$ the ring inside $\mathcal{K}[\![Z, Z^{-1}]\!]$ consisting of those elements converging for $z \in \mathbb{C}_p$ with absolute value in I. For fixed $r \in (0,1)$ we then set

$$\mathcal{R}^{[r,1)}_{\mathcal{K}} := \varprojlim_{r<s<1} \mathcal{R}^{[r,s]}_{\mathcal{K}}$$

and

$$\mathcal{R}_{\mathcal{K}} := \bigcup_{0<r<1} \mathcal{R}_{\mathcal{K}}^{[r,1)}.$$

Recall from [Col16, p. 10] that we have $\mathcal{R}_{\mathcal{K}} = \mathcal{R}_{\mathcal{K}}^{+} \cap \mathcal{K}[\![Z]\!]$. We do also have a φ- and a ψ-operator on $\mathcal{R}_{\mathcal{K}}$ which we again denote by φ_L and ψ_L respectively (cf. [Col16, p. 11]). They clearly restrict to endomorphisms of $\mathcal{R}_{\mathcal{K}}^{+}$ and fulfill a projection formula similar to one from Remark 4.2.3. Let $\Omega \in \mathbb{C}_p$ be the period of a fixed generator t' of the dual of the Tate module $\mathcal{T}\mathcal{G}_{\phi}$. This means, that the power series attached to t' starts with $\Omega X + \cdots$ (cf. [ST01, p. 457]) and for $b \in \mathcal{O}_L$ we set (cf. [Col16, p. 9])

$$\eta(b, Z) := \exp(b\Omega \log_{\mathrm{LT}}(Z)).$$

From [Col16, p. 9] we then also deduce $\eta(b, Z) \in \mathcal{O}_{\mathbb{C}_p}[\![Z]\!]^{\times}$. By abuse of notation, we also write $\eta(b, Z)$ for $b \in \mathcal{O}_L/\pi_L\mathcal{O}_L$ instead of $\eta(a, Z)$ where $a \in \mathcal{O}_L$ is a lift of b. Note that η and ψ_L have the following correlation (cf. [Col16, p. 11])

$$\psi_L(\eta(b, Z)) = \begin{cases} \eta\left(\frac{b}{\pi_L}, Z\right), & \text{if } b \in \pi_L\mathcal{O}_L \\ 0, & \text{else.} \end{cases}$$

If $\Omega \in \mathcal{K}$ we deduce from [Col16, p. 11]

$$\mathcal{R}_{\mathcal{K}} = \bigoplus_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b, Z)\varphi_L(\mathcal{R}_{\mathcal{K}}).$$

In particular, for $x \in \mathcal{R}_{\mathcal{K}}$ we then have

$$x = \sum_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b, Z)\varphi_L(\psi_L(\eta(-b, Z)x)).$$

Since $\eta(b, Z)$ is invertible in $\mathcal{K}[\![Z]\!]$ and both, φ_L and ψ_L are given by power series, the above decomposition holds true for $\mathcal{R}_{\mathcal{K}}^{+}$, i.e. if $\Omega \in \mathcal{K}$ we have

$$\mathcal{R}_{\mathcal{K}}^{+} = \bigoplus_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b, Z)\varphi_L(\mathcal{R}_{\mathcal{K}}^{+}).$$

Furthermore, recall from [SV19, p. 34] that we have an isomorphism

$$(\mathcal{R}_{\mathcal{K}}^{+})^{\psi_L = 0} \cong D_L(\Gamma_L, \mathcal{K}).$$

Finally, by $\varphi_L \otimes \varphi_{S_\infty}$ and $\psi_L \otimes \psi_{S_\infty}$ respectively we then also have a φ- and a ψ-operator on $\mathcal{R}_{\mathcal{K}}^{+} \otimes_{\mathcal{O}_F} S_\infty$. These endomorphisms then extend by continuity to

$\mathcal{R}_{\mathcal{K}}^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty$. On $\mathcal{R}_{\mathcal{K}}^{+}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$ we then clearly can make a similar construction for a φ- and ψ-operator.

Lemma 6.6.2.
Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$ and $V = T[1/\pi_L]$. Then we have a homomorphism

$$\begin{aligned}\left(\varphi^*\mathcal{N}_{F_\infty|L}(V)\right)^{\psi=0} &\longrightarrow \left(\mathcal{R}_F^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi_L=0} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V) \longrightarrow \cdots \\ \cdots &\longrightarrow \left(\mathcal{R}_{\mathbb{C}_p}^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi_L=0} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V).\end{aligned}$$

Proof.
From [SV19, Corollary 1.14, p. 14–15] we deduce, that there is a homomorphism

$$\mathcal{N}_L(V) \to \mathcal{R}_L^{+} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V),$$

which by tensoring with $\mathcal{O}_F$ induces a homomorphism

$$\mathcal{N}_{F|L}(V) = \mathcal{O}_F \otimes_{\mathcal{O}_L} \mathcal{N}_L(V) \to \mathcal{O}_F \otimes_{\mathcal{O}_L} \mathcal{R}_L^{+} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V) = \mathcal{R}_F^{+} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V).$$

Together with Proposition 6.5.10 this induces a homomorphism

$$\mathcal{N}_{F_\infty|L}(V) \cong S_\infty\widehat{\otimes}_{\mathcal{O}_F}\mathcal{N}_{F|L}(V) \to S_\infty\widehat{\otimes}_{\mathcal{O}_F}\left(\mathcal{R}_F^{+} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V)\right).$$

But since $\mathcal{D}_{\mathrm{cris},L}(V)$ is a finite dimensional L-vector space we have

$$S_\infty\widehat{\otimes}_{\mathcal{O}_F}\left(\mathcal{R}_F^{+} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V)\right) \cong \left(S_\infty\widehat{\otimes}_{\mathcal{O}_F}\mathcal{R}_F^{+}\right) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V)$$

and therefore we can show with exactly the same proof as in loc. cit. that there is a homomorphism

$$\left(\varphi^*\left(\mathcal{N}_{F|L}(V)\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)\right)^{\psi=0} \to \left(\mathcal{R}_F^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi_L=0} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V).$$

Since S_∞ is torsion free as $\mathcal{O}_F$-module, it is flat and therefore we get an inclusion

$$\left(\mathcal{R}_F^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi_L=0} \hookrightarrow \left(\mathcal{R}_{\mathbb{C}_p}^{+}\widehat{\otimes}_{\mathcal{O}_F} S_\infty\right).^{\psi_L=0}$$

Together with the above homomorphism, this then gives us the desired homomorphism. $\square$

Lemma 6.6.3.
The elements of $\mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$ *can be written in the form*

$$\sum_{b\in\mathcal{O}_L/\pi_L\mathcal{O}_L}\eta(b,Z)\varphi_L(x_b)$$

with $x_b \in \mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$.

Proof.
It suffices to prove the claim for $\mathcal{R}^+_{\mathbb{C}_p}\otimes_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$, the statement for the completion then follows by continuity.
So let $x \in \mathcal{R}^+_{\mathbb{C}_p}\otimes_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$ and write $x = \sum_{i=1}^m x_i\otimes y_i$ for some $x_i \in \mathcal{R}^+_{\mathbb{C}_p}$ and $y_i \in \Lambda_{\mathcal{O}_F}(\Upsilon)$. Then, for every $1\le i\le m$ there exist $x_b^{(i)} \in \mathcal{R}^+_{\mathbb{C}_p}$ (cf. the discussion above) with $b\in\mathcal{O}_L/\pi_L\mathcal{O}_L$ such that

$$x_i = \sum_{b\in\mathcal{O}_L/\pi_L\mathcal{O}_L}\eta(b,Z)\varphi_L(x_b^{(i)}).$$

We then compute

$$\begin{aligned}
x &= \sum_{i=0}^m\left(\sum_{b\in\mathcal{O}_L/\pi_L\mathcal{O}_L}\eta(b,Z)\varphi_L\left((x_b^{(i)})\right)\right)\otimes y_i\\
&= \sum_{b\in\mathcal{O}_L/\pi_L\mathcal{O}_L}\eta(b,Z)\left(\sum_{i=0}^m\varphi_L\left(x_b^{(i)}\right)\otimes y_i\right)\\
&= \sum_{b\in\mathcal{O}_L/\pi_L\mathcal{O}_L}\eta(b,Z)\varphi_L\left(\sum_{i=0}^m x_b^{(i)}\otimes y_i'\right),
\end{aligned}$$

where $y_i' \in \Lambda_{\mathcal{O}_F}(\Upsilon)$ is a preimage of y_i for every $1\le i\le m$ under the Frobenius on $\Lambda_{\mathcal{O}_F}(\Upsilon)$ (which is bijective cf. Remark 6.4.17 and Proposition 6.4.18). □

As described at the beginning of this section, the following Lemma is the main difference to the construction of the regulator map in [LZ14a].

Lemma 6.6.4.
We have

$$\left(\mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}S_\infty\right)^{\psi_L=0} \cong \left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0}\widehat{\otimes}_{\mathcal{O}_F}S_\infty.$$

Proof.

With the above discussion about the decomposition

$$\mathcal{R}^+_{\mathbb{C}_p} = \bigoplus_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b,Z)\varphi_L(\mathcal{R}^+_{\mathbb{C}_p}),$$

this proof is nearly analogous to the proof of [LZ14a, Proposition 3.12, p. 12].
In Proposition 6.4.18 we saw that S_∞ is a free rank one module of $\Lambda_{\mathcal{O}_F}(\Upsilon)$, say with basis ξ. Since

$$\mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}S_\infty \cong \left(\mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)\right) \otimes_{\mathcal{R}^+_{\mathbb{C}_p}\otimes_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)} \left(\mathcal{R}^+_{\mathbb{C}_p}\otimes_{\mathcal{O}_F} S_\infty\right)$$

(cf. [Mat70, Theorem 55, p. 170]) the elements of $\mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}S_\infty$ are of the form $x(1\otimes\xi)$ with $x \in \mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$. Then for every $b \in \mathcal{O}_L/\pi_L\mathcal{O}_L$ let $x_b \in \mathcal{R}^+_{\mathbb{C}_p}\widehat{\otimes}_{\mathcal{O}_F}\Lambda_{\mathcal{O}_F}(\Upsilon)$ such that

$$x = \sum_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b,Z)\varphi_L(x_b),$$

(cf. Lemma 6.6.3). Applying ψ_L to such an element then gives us

$$\begin{aligned}
\psi_L(x(1\otimes\xi)) &= \psi_L\left(\sum_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \eta(b,Z)\varphi_L(x_b)(1\otimes\xi)\right) \\
&= \sum_{b \in \mathcal{O}_L/\pi_L\mathcal{O}_L} \psi_L(\eta(b,Z))x_b(1\otimes\psi_{S_\infty}(\xi)) \\
&= \eta(0,Z)x_0(1\otimes\psi_{S_\infty}(\xi)).
\end{aligned}$$

Since ψ_{S_∞} is an isomorphism on S_∞ (cf. Remark 6.4.17), we deduce that for $\psi_L(x(1\otimes\xi)) = 0$ it must be $x_0 = 0$. Therefore we have

$$x = \sum_{b \in (\mathcal{O}_L/\pi_L\mathcal{O}_L)^\times} \eta(b,Z)\varphi_L(x_b),$$

i.e. $x(1\otimes\xi) \in \left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0}\widehat{\otimes}_{\mathcal{O}_L}S_\infty$. □

Lemma 6.6.5.
We have an injective homomorphism

$$\left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0}\widehat{\otimes}_{\mathcal{O}_F}S_\infty \hookrightarrow D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F}D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p).$$

Proof.

From the discussion before Lemma 6.6.2 we deduce the isomorphism

$$\left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0} \cong D_L(\Gamma_L, \mathbb{C}_p)$$

As in [LZ14a, p. 15] we have a continuous inclusion

$$S_\infty \hookrightarrow \Lambda_{\widehat{\mathcal{O}}_{F_\infty}}(\Upsilon) \hookrightarrow D_{\mathbb{Q}_p}(\Upsilon, \widehat{F_\infty}).$$

Since $\mathcal{R}^+_{\mathbb{C}_p}$ is torsion free as $\mathcal{O}_F$-modules, $\left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0}$ is also torsion free as $\mathcal{O}_F$-module. So in particular, it is flat and we get an inclusion

$$\left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0} \otimes_{\mathcal{O}_F} S_\infty \hookrightarrow \left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0} \otimes_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \widehat{F_\infty}).$$

Since projective limits are left exact, this inclusion extends to the completion and so we get the desired inclusion by composing the above homomorphisms

$$\left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0} \widehat{\otimes}_{\mathcal{O}_F} S_\infty \hookrightarrow \left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi_L=0} \widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \widehat{F_\infty}) \xrightarrow{\cong} \cdots$$
$$\cdots \xrightarrow{\cong} D_L(\Gamma_L, \mathbb{C}_p) \widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \widehat{F_\infty}) \hookrightarrow D_L(\Gamma_L, \mathbb{C}_p) \widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \mathbb{C}_p).$$

□

Definition 6.6.6.
Let $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_L)$ and $V = T[1/\pi_L]$ such that T has no quotient isomorphic to the trivial representation. We define the regulator map $\mathcal{L}^{\Gamma_L,\Upsilon}_V$ as the composite of the above discussed maps:

$$\begin{aligned}
H^1_{\mathrm{Iw}}(F_\infty L_\infty | L, T) &\xrightarrow{\cong} \mathcal{N}_{F_\infty|L}(T(\tau^{-1}))^{\psi=1} \\
&\longrightarrow \left(\varphi^* \mathcal{N}_{F_\infty|L}(V(\tau^{-1}))\right)^{\psi=0} \\
&\longrightarrow \left(\mathcal{R}^+_F \widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi=0} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \\
&\longrightarrow \left(\mathcal{R}^+_{\mathbb{C}_p} \widehat{\otimes}_{\mathcal{O}_F} S_\infty\right)^{\psi=0} \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \\
&\longrightarrow \left(\mathcal{R}^+_{\mathbb{C}_p}\right)^{\psi=0} \widehat{\otimes}_{\mathcal{O}_F} S_\infty \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \\
&\hookrightarrow D_L(\Gamma_L, \mathbb{C}_p) \widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon, \mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})).
\end{aligned}$$

Following the order of appearance above, these maps are subject of Lemma 6.6.1, Remark 6.5.5, Lemma 6.6.2 (line 3 and 4), Lemma 6.6.4 and Lemma 6.6.5 respectively.

The following theorem is the analogue of [LZ14a, Theorem 4.7, p. 16–17] adapted to our situation.

Theorem 6.6.7.
Let $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_L)$ and $V = T[1/\pi_L]$ such that T has no quotient isomorphic to the trivial representation. Then the regulator map $\mathcal{L}_V^{\Gamma_L,\Upsilon}$ from the above Definition 6.6.6 is a homomoprhism of $\Lambda_{\mathcal{O}_F}(\Gamma_L \times \Upsilon)$-modules and has the following two properties which uniquely determine this homomorphism:

1. *If $E|F$ is a finite, unramified extension contained in $F_\infty K_\infty$, we get a commutative diagram*

$$\begin{tikzcd} H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon}"] \arrow[d] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \arrow[d] \\ H^1_{\mathrm{Iw}}(EL_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon_E}"] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon_E,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})), \end{tikzcd}$$

   where $\Upsilon_E = \mathrm{Gal}(E|F)$ and $\mathcal{L}_V^{\Gamma_L,\Upsilon_E}$ is defined in the same way as $\mathcal{L}_V^{\Gamma_L,\Upsilon}$.

2. *For $x \in H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T)$ and a character $\eta\colon \Gamma_L \to \mathbb{C}_p$ the function $\omega \mapsto \mathcal{L}_V^{\Gamma_L,\Upsilon}(x)(\eta\otimes\omega)$ is a bounded $\mathbb{Q}_p$-analytic function.*

Proof.
The proof of [LZ14a, Theorem 4.7, p. 16–17] translates to our situation. □

The goal now is to establish an analogous result to [LZ14a, Theorem 4.13, p. 20]. To do this, we adapt the relevant statements from [LZ14a] to our situation. The proofs in our situation are all analogous and we only cite the relevant parts from [LZ14a]. To simplify a comparison, we cite all the statements of [LZ14a] which have an input to [LZ14a, Theorem 4.13, p. 20] and translate them into our situation.

Lemma 6.6.8.
Let $T \in \mathbf{Rep}^{\mathrm{cris,an}}_{\mathcal{O}_L}(G_L)$ and let M be a finitely free $\mathcal{O}_L$-module with a continuous action from Υ via a homomorphism $\Lambda_{\mathcal{O}_L}(\Upsilon) \to \mathrm{End}_{\mathcal{O}_L}(M)$. We then have canonical isomorphisms

$$\mathcal{N}_{F|L}(T)\otimes_{\mathcal{O}_F}\left(\widehat{\mathcal{O}}_{F_\infty}\otimes_{\mathcal{O}_L} M\right)^{\Upsilon} \cong \mathcal{N}_{F|L}(T\otimes_{\mathcal{O}_L} M)$$

and

$$M\otimes_{\mathcal{O}_L}\mathcal{N}_{F_\infty|L}(T)\cong \mathcal{N}_{F_\infty|L}(M\otimes_{\mathcal{O}_L} T).$$

Proof.
Since taking Wach Modules is a $\otimes$-functor (cf. [KR09, Corollary (3.3.8), p. 460]), the first isomorphism is obtained exactly as in [LZ14a, Proposition 3.13, p. 13]. The second is [LZ14a, Theorem 3.15, p. 13]. □

Proposition 6.6.9.
Let $X|L$ be a finite extension and $\omega\colon \Upsilon \to \mathcal{O}_X^\times$ a continuous homomorphism. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_X \otimes_{\mathcal{O}_L} H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon}"] \arrow[d] & X \otimes_L D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \arrow[d] \\
H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T(\omega)) \arrow[r, "\mathcal{L}_{V(\omega)}^{\Gamma_L,\Upsilon}"] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1}\omega))
\end{tikzcd}$$

Proof.
This is the same proof as the one of [LZ14a, Proposition 4.12, p. 19]. □

Corollary 6.6.10.
Let $X|L$ and $E|F$ be finite extensions such that E is unramified and contained in $F_\infty K_\infty$ and let $\omega\colon \Upsilon \to \mathcal{O}_X^\times$ be a continuous homomorphism. Then the following diagram commutes

$$\begin{tikzcd}
H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon}"] \arrow[d] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \arrow[d] \\
H^1_{\mathrm{Iw}}(EL_\infty|L,T(\omega)) \arrow[r, "\mathcal{L}_{V(\omega)}^{\Gamma_L,\Upsilon_E}"] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon_E,\mathbb{C}_p) \otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1}\omega))
\end{tikzcd}$$

Proof.
As mentioned before [LZ14a, Theorem 4.13, p. 20] this now is an immediate consequence of Theorem 6.6.7 and Proposition 6.6.9. Precisely we obtain the following

commutative diagram

$$\begin{tikzcd}
H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon}"] \arrow[d] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \arrow[d] \\
\mathcal{O}_X \otimes_{\mathcal{O}_L} H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T) \arrow[r, "\mathcal{L}_V^{\Gamma_L,\Upsilon}"] \arrow[d] & X\otimes_L D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1})) \arrow[d] \\
H^1_{\mathrm{Iw}}(F_\infty L_\infty|L,T(\omega)) \arrow[r, "\mathcal{L}_{V(\omega)}^{\Gamma_L,\Upsilon}"] \arrow[d] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1}\omega)) \arrow[d] \\
H^1_{\mathrm{Iw}}(EL_\infty|L,T(\omega)) \arrow[r, "\mathcal{L}_{V(\omega)}^{\Gamma_L,\Upsilon_E}"] & D_L(\Gamma_L,\mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F} D_{\mathbb{Q}_p}(\Upsilon_E,\mathbb{C}_p)\otimes_L \mathcal{D}_{\mathrm{cris},L}(V(\tau^{-1}\omega)),
\end{tikzcd}$$

where the vertical maps in the upper square send an element x to $1\otimes x$. Therefore the upper square commutes evidently. The middle square commutes because of Proposition 6.6.9 and the latter because of Theorem 6.6.7. □

List of Symbols

A^δ

$= \bigcup_{U \leq G \text{ open}} A^U$, for an abelian group A with an action from a profinite group G. 24

$\mathbf{Ab}$

Category of abelian groups. 26

$\mathcal{ABS}_G$

Category of (abstract) abelian groups with an action from a profinite group G together with group homomorphisms respecting the action from G. 23

$\mathcal{ABS}_{GM}$

Category of (abstract) abelian groups with commuting actions from a profinite group G and a topological monoid M together with group homomorphisms respecting the actions from G and M. 23

$\mathcal{ABS}_M$

Category of (abstract) abelian groups with an action from a topological monoid M together with group homomorphisms respecting the action from M. 23

$\mathbf{A}^0_{\mathrm{cris}}$

$= \left\{ \sum_{n=0}^N a_n \frac{\xi^n}{n!} \,\middle|\, N \in \mathbb{N}_0, a_n \in W(\mathcal{O}_{\mathbb{C}_p^\flat}) \right\}$. 172

$\mathbf{A}_{\mathrm{cris}}$

$= \varprojlim_n \mathbf{A}^0_{\mathrm{cris}} / p^n \mathbf{A}^0_{\mathrm{cris}}$. 172

$\mathbf{A}_{\mathrm{cris},L}$

$= \mathbf{A}_{\mathrm{cris}} \otimes_{=_{L_0}} \mathcal{O}_L$. 172

Ad

the action of G on $C^{\bullet}_{\mathrm{cts}}(H,M)$ given by

$$\mathrm{Ad}(g)(c)(h_0,\dots,h_n) = g(c(g^{-1}h_0g,\dots,g^{-1}h_ng)),$$

where $c \in C^n_{\mathrm{cts}}(H,M)$, G is a profinite group, $H \triangleleft G$ a closed, normal subgroup and M a discrete G-module
It denotes also the induced action from G/H on $\mathbf{R}\Gamma_{\mathrm{cts}}(H,M)$. 145

$\widetilde{\mathrm{Ad}}$

the action of G on $\mathrm{Ind}^G_U(M)$ given by $\widetilde{\mathrm{Ad}}(g)(f)(\sigma) = g(f(g^{-1}\sigma g))$ of G/U, where G is a profinite group, $U \triangleleft G$ an open, normal subgroup, M a discrete $\mathcal{O}_L[G]$-module, $f \in \mathrm{Ind}^G_U(M)$ and $g \in G$. This action is trivial on U and therefore induces an action of G/U.
It denotes also the action $\widetilde{\mathrm{Ad}}(gU)(f)(\sigma U) = f(\sigma g U)$ from G/U on ${}_UM$. If $H \triangleleft G$ is a closed normal subgroup, such that G/H is abelian, then it also denotes the induced actions from both, G/H and $\mathcal{O}_L[\![G/H]\!]$, on $F_{G/H}(M)$.
Furthermore, it denotes the action $\widetilde{\mathrm{Ad}}gU(a\otimes xU) = a\otimes xg^{-1}U$ on $T\otimes\mathcal{O}_L[G_K/U]$. 143

ad_M

The homomorphism

$$\mathbf{A} \otimes_{\mathcal{O}_L} \mathcal{V}_{\mathcal{K}}(M) \to \mathbf{A} \otimes_{\mathbf{A}_{\mathcal{K}}} M,\ a \otimes v \mapsto av$$

for $M \in \mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{\mathcal{K}})$ and $\mathcal{K}|L$ finite. 76

ad_V

The homomorphism

$$\mathbf{A} \otimes_{\mathbf{A}_{\mathcal{K}}} \mathcal{M}_{\mathcal{K}}(V) \to \mathbf{A} \otimes_{\mathcal{O}_L} V,\ a \otimes m \mapsto am$$

for $V \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_{\mathcal{K}})$ and $\mathcal{K}|L$ finite. 76

$\mathscr{A}_L$

$= \varprojlim_n \mathcal{O}_L/\pi_L^n\mathcal{O}_L((X))$; in Chapter 4 we use the variable Z. 54

$\mathbf{A}_L$

Image of the inclusion $\mathscr{A}_L \hookrightarrow W(\mathbf{E}_L)_L,\ X \mapsto \omega_\phi$. 56

$\mathbf{B}_{\mathrm{dR}}$

$= \mathbf{B}_{\mathrm{dR}}^{+}[1/\xi]]$ - de Rham period ring. 171

$\mathbf{B}_{\mathrm{dR}}^{+}$

$= \varprojlim_n W(\mathcal{O}_{\mathbb{C}_p^\flat})[1/p]/(\xi)^n$. 171

$\mathscr{B}_L$

Fraction field of $\mathscr{A}_L$. 94

$\mathbf{B}_L$

Quotient field of $\mathbf{A}_L$. 58

$\mathbf{B}_L^{\mathrm{nr}}$

Quotient field of $\mathbf{A}_L^{\mathrm{nr}}$. 58

$\mathbf{B}$

Quotient field of $\mathbf{A}$. 58

$\mathbf{B}_{\mathcal{K}|L}$

Quotient field of $\mathbf{A}_{\mathcal{K}|L}$ for $\mathcal{K}|L$ finite. 58

$\mathscr{B}_{\mathcal{K}|L}$

Fraction field of $\mathscr{A}_{\mathcal{K}|L}$, for $\mathcal{K}|L$ finite, unramified. 94

$C^{X\text{-}\mathsf{an}}(B, E)$

locally X-analytic functions from B to E, where $X|\mathbb{Q}_p$ is finite and $E|X$ is complete, W is a finite dimensional X-vector space, and $B \subseteq W$ is a closed polydisk. 174

$C^\bullet[n]$

shift of the complex by $n \in \mathbb{Z}$, i.e. we have $C^i[n] = C^{i+n}$. 40

$\mathcal{C}_X^\bullet(G, A)$

Total complex of the double complex $\mathcal{C}^\bullet(G, A) \xrightarrow{X-1} \mathcal{C}^\bullet(G, A)$. 30

$\mathcal{C}_f^\bullet(G, A)$

$= \mathcal{C}_X^\bullet(G, A)$ where the $\mathbb{N}_0$-action on A comes from an endomorphism f of A.. 31

$\mathcal{DIS}_M$

Category of discrete abelian groups with a continuous action from a topological monoid M together with continuous group homomorphisms respecting the action from M. 22

Δ_1

Action from Υ_n on $\mathcal{O}_{F_n}$ with

$$\Delta_1(h, \sum x_g \cdot g) = \sum h(x_g) \cdot g$$

. 177

Δ_2

Action from Υ_n on $\mathcal{O}_{F_n}$ with

$$\Delta_1(h, \sum x_g \cdot g) = \sum x_{h^{-1}g} \cdot g$$

. 177

$\mathrm{d}^n_{\mathrm{Tot}(A^{\bullet,\bullet})}$

n-th differential of the total complex $\mathrm{Tot}(A^{\bullet,\bullet})$ of the double complex $A^{\bullet,\bullet}$. 30

∂_{cts}

the differential $X^{n-1}_{\mathrm{cts}}(G, A) \to X^n_{\mathrm{cts}}(G, A)$. 13

∂_{inv}

Invariant derivation corresponding to $\mathrm{dlog}_{\mathrm{LT}}$. 84

$D_X(G, E)$

E-valued locally X-analytic distributions on B, where $X|\mathbb{Q}_p$ is finite and $E|X$ is complete, and G is a Lie group over X. Equivalently, this is the continuous dual of $C^{X\text{-an}}(G, E)$. 174

$\mathcal{D}(-)$

$= \mathrm{Hom}_{\mathbf{A}_{\mathcal{K}|L}}(-, \Omega^1_{\mathbf{A}_{\mathcal{K}|L}} \otimes_{\mathbf{A}_{\mathcal{K}|L}} \mathbf{B}_{\mathcal{K}|L}/\mathbf{A}_{\mathcal{K}|L})$, where $\mathcal{K}|L$ is finite. 162

$\overline{\mathrm{D}_{\mathcal{K}}}(-)$

$= \mathrm{Hom}_{\Lambda_{\mathcal{K}}}(-, \Lambda^\vee_{\mathcal{K}})$, so called Matlis dual, where $\mathcal{K}|L$ finite. 153

f_M^{lin}

Linearization of the f-linear endomorphism f_M of M, where M is an R-module and f is an endomorphism of R, i.e. the homomorphism

$$f_M^{\mathrm{lin}} \colon R\ {}_f\!\otimes_R M \to M,\ a \otimes m \mapsto a f_M(m)$$

. 74

F_n

Unramified extension of degree p^n over F, where $F|L$ is unramified. 169

F_∞

$= \cup_n F_n$. 169

Fr

Frobenius on $W(\mathbb{C}_p^\flat)_L$. 56

f^ϑ

Applying ϑ to the coefficients of $f \in \mathbf{A}_{\mathcal{K}}$, where $\mathcal{K}|L$ is finite unramified and ϑ is an $\mathcal{O}_L$-linear endomorphism of $\mathcal{O}_{\mathcal{K}}$, i.e.

$$f^\vartheta = \sum \vartheta(a_i)\omega_\phi^i$$

. 71

Γ_K

$= \mathrm{Gal}(K_\infty|K)$. 53

Γ_L

$= \mathrm{Gal}(L_\infty|L)$. 53

H_K

$= \mathrm{Gal}(\overline{\mathbb{Q}_p}|K_\infty)$. 53

H_L

$= \mathrm{Gal}(\overline{\mathbb{Q}_p}|L_\infty)$. 53

$G_{\mathcal{K}}$

Absolute Galois Group of $\mathcal{K}|\mathbb{Q}_p$. 51

g_{LT}

The inverse of $\frac{\partial(\mathcal{G}_\phi(X,Y))}{\partial Y}|_{(X,Y)=(0,Z)}$ in $\mathcal{O}_L[\![Z]\!]$. 84

$\mathcal{G}_{\phi,n}$

$= \ker([\pi_L^n]_\phi \colon \mathfrak{M} \to \mathfrak{M}) = \{x \in \mathfrak{M} \mid [\pi_L^n]_\phi(x) = 0\}$. 52

$[a]_\phi$

Endomorphism of $\mathcal{G}_\phi$ associated to $a \in \mathcal{O}_L$. 52

$\mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(V)$

$= \mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1}\mathcal{D}_{\mathrm{dR}}(V)$, where $V \in \mathbf{Rep}_L^{(\mathrm{fg})}(G_L)$. 173

g_{u,t_0}

Unique Laurent series in $(\mathcal{O}_K((Z))^\times)^{\mathcal{N}=\mathrm{id}}$ to $u = (u_n)_n \in \varprojlim_n K_n^\times$ with $\sigma_{K|L}^{-n}(g_{u,t_0}(t_{0,n})) = u_n$ were $t_0 = (t_{0,n})_n$ is an $\mathcal{O}_L$-generator of $\mathcal{T}\mathcal{G}_\phi$. 90

$H^*_{\mathrm{Iw}}(K_\infty|K,-)$

Generalized Iwasawa cohomology, i.e. for $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ we have

$$H^i_{\mathrm{Iw}}(K_\infty|K,V) = \varprojlim_{\substack{K\subseteq E\subseteq K_\infty \\ \text{finite}}} H^i(G_E,V)$$

where $i \in \mathbb{N}_0$. 109

$\mathrm{Hom}^{\mathrm{cts}}$

Continuous homomorphisms. 97

$\mathrm{Hom}_R(M,N)$

R-linear Homomorphisms between the R-modules M and N. 21

$\mathcal{H}^*_X(G,A)$

Cohomology of the complex $\mathcal{C}^\bullet_X(G,A)$. 30

$\mathcal{H}^*_f(G,A)$

Cohomology of the complex $\mathcal{C}^\bullet_f(G,A)$. 31

$\mathrm{im}(f)$

Image of the homomorphism f. 9

$\mathrm{Ind}_U^G(M)$

$= \{f\colon G \to X \mid f$ locally constant and U-linear$\}$, for a profinite group G an open subgroup U and a discrete U-module M.. 141

ι

the involution $x \mapsto x^{-1}$ of a group. 150

$\mathcal{K}^\flat$

Tilt of the perfectoid field $\mathcal{K} \subseteq \mathbb{C}_p$. 52

$\ker(f)$

Kernel of the homomorphism f. 9

$k_{\mathcal{K}}$

Residue class field of the finite extension $\mathcal{K}|\mathbb{Q}_p$. 51

K_n

$= K(\mathcal{G}_\phi[\pi_L^n]) = KL_n$. 53

K_∞

$= \cup_n K_n$. 53

$\mathcal{K}^{\mathrm{ur}}$

Maximal unramified extension of $\mathbb{Q}_p$ inside $\mathcal{K}$. 51

$\Lambda_R(G)$

$= \varprojlim_{H \triangleleft G \text{ open}} R(G/H)$ the Iwasawa module of the profinite group G with coefficients in the ring R. 170

$\Lambda_{\mathcal{K}}$

$= \mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]$, Iwasawa algebra of $\Gamma_{\mathcal{K}}$, with $\mathcal{K}|L$ finite. 141

$\mathcal{L}_V^{\Gamma_L,\Upsilon}$

Regulator map from $H^1_{\mathrm{Iw}}(K_\infty F_\infty|K, T)$ to $D_L(\Gamma_L, \mathbb{C}_p)\widehat{\otimes}_{\mathcal{O}_F}(\Upsilon, \mathbb{C}_p)\otimes_L \mathcal{M}_{\mathrm{cris},L}(V)$, where $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$ and $V = T[1/\pi_L]$. 198

$\varprojlim^r$

r-th right derived functor of $\varprojlim$. 46

L_n

$= L(\mathcal{G}_\phi[\pi_L^n])$. 53

L_∞

$= \cup_n L_n$. 53

$\mathrm{Map}_{\mathrm{cts}}(X, Y)$

Set of continuous maps between the topological spaces X and Y. 9

$\mathrm{M}(K, U)$

Set of all $f \in \mathrm{Map}_{\mathrm{cts}}(X, Y)$ with $f(K) \subseteq U$, where $K \subseteq X$ compact and $U \subseteq Y$ open, i.e. basis open sets for the compact open topology on $\mathrm{Map}_{\mathrm{cts}}(X, Y)$. 9

$\mathcal{D}_{\mathrm{dR}}(-)$

$= \big(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} -\big)^{G_L}$. 173

M^ι

the Λ_K-module M where Γ_K acts via the involution, i.e. we have $\gamma \cdot m = \gamma^{-1} m$ for all $\gamma \in \Gamma_K$ and $m \in M$. 150

$\mathcal{M}_{\mathcal{K}|L}$

Functor from $\mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_{\mathcal{K}})$ to $\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{\mathcal{K}|L})$ with $\mathcal{M}_{\mathcal{K}|L}(V) = (\mathbf{A} \otimes_{\mathcal{O}_L} V)^{H_{\mathcal{K}}}$; defining an equivalence with inverse $\mathcal{V}_{\mathcal{K}|L}$ ($\mathcal{K}|L$ finite). 75

$\mathbf{Mod}^{\text{ét}}_{\varphi,\Gamma}(\mathbf{A}_{\mathcal{K}|L})$

Category of étale $(\varphi_{\mathcal{K}|L}, \Gamma_{\mathcal{K}})$-modules over $\mathbf{A}_{\mathcal{K}|L}$, where $\mathcal{K}|L$ is finite. 74

$\mathbf{Mod}^{\mathrm{an}}_{\varphi,\Gamma}(\mathbf{A}^+_{\mathcal{K}|L})$

Category of analytic $(\varphi_{\mathcal{K}|L}, \Gamma_{\mathcal{K}})$-modules over $\mathbf{A}^+_{\mathcal{K}|L}$, where $\mathcal{K}|L$ is finite. 173

$\mathbf{Mod}^{\text{ét},f}_{\varphi,\Gamma}(\mathbf{A}_{\mathcal{K}|L})$

Category of free étale $(\varphi_{\mathcal{K}|L}, \Gamma_{\mathcal{K}})$-modules over $\mathbf{A}_{\mathcal{K}|L}$, where $\mathcal{K}|L$ is finite. 169

${}_U M$

$= \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G/U], M)$, where M is an ind-admissible $\mathcal{O}_L[G]$-module, G a profinite group and $U \subseteq G$ an open subgroup. 141

$\mu_{\pi_L^n}$

Multiplication with π_L^n on an $\mathcal{O}_L$-module. 137

$M^\vee$

$= \mathrm{Hom^{cts}}_{\mathcal{O}_L}(M, L/\mathcal{O}_L)$. Pontrjagin dual of the $\mathcal{O}_L$-module M. 108

$\mathcal{N}_{\mathcal{K}|L}(T)$

Wach module of $T \in \mathbf{Rep}_{\mathcal{O}_L}^{\text{cris,an}}(G_{\mathcal{K}})$ for $\mathcal{K}|L$ finite. 189

$\mathcal{N}_{F_\infty|L}(T)$

$= \varprojlim_n \mathcal{N}_{F_n|L}(T)$. 190

$\mathbb{N}$

Natural numbers, starting with 1. 9

$\mathbb{N}_0$

Natural numbers, starting with 0, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. 9

$\mathrm{Nor}_{K|L}$

$= \phi_{K|L}^{-1} \circ \mathrm{Nor}$, where Nor is the norm map of $\mathscr{B}_{K|L}|\phi_{K|L}(\mathscr{B}_{K|L})$. 95

$\widetilde{\mathcal{N}}$

Unique multiplicative map from $\mathcal{O}_K[\![Z]\!]$ to itself with

$$\left(\varpi_{K|L} \circ \widetilde{\mathcal{N}}\right)(f(Z)) = \prod_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z)$$

for all $f \in \mathcal{O}[\![Z]\!]$. 87

$\mathcal{N}$

$= \sigma_{K|L}^{-1} \circ \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}} \circ \sigma_{K|L}^{-1}$. 88

$\mathcal{O}_{\mathcal{K}}$

Ring of integers of the extension $\mathcal{K}|\mathbb{Q}_p$. 51

$\Omega^1_{\mathscr{A}_{\mathcal{K}|L}}$

$= \mathscr{A}_{\mathcal{K}|L}\mathrm{d}Z$; free rank one differential forms over $\mathscr{A}_{\mathcal{K}|L}$, for $\mathcal{K}|L$ finite, unramified. 96

ω

Uniformizer of $\mathbf{E}_L$. 55

ω_ϕ

Lift of $\omega \in \mathbf{E}_L$ to $W(\mathbf{E}_L)_L$. 56

$\varphi^*\mathcal{N}_{\mathcal{K}|L}(V)$

The $\mathbf{A}^+_{\mathcal{K}|L}$-submodule of $\mathcal{N}_{\mathcal{K}|L}(V)[1/Q_\phi]$ generated by $\mathrm{im}(\varphi_{\mathcal{N}_{\mathcal{K}}(V)})$, where $V \in \mathbf{Rep}_L^{\mathrm{cris,an}}$ and $\mathcal{K}|L$ finite. 189

φ_{S_∞}

$= \varprojlim_n \Delta_1(\sigma_{F_n})$, Frobenius of S_∞. 186

$\pi_{\mathcal{K}}$

Prime element of the finite extension $\mathcal{K}|\mathbb{Q}_p$. 51

$\widetilde{\pi_L}$

$= (\pi_n \bmod \pi_L \mathcal{O}_{\mathbb{C}_p})_n \in \mathcal{O}_{\mathbb{C}_p^\flat}$, where $\pi_0 = \pi_L$ and $\pi_{n+1}^{q_L} = \pi_n$. 171

$\widetilde{\psi_{\mathrm{Col}}}$

Unique endomorphism of $\mathcal{O}_K[\![Z]\!]$ with

$$\left(\varpi_{K|L} \circ \widetilde{\psi_{\mathrm{Col}}}\right)(f(Z)) = \sum_{a \in \mathcal{G}_\phi[\pi_L]} f(a +_{\mathcal{G}_\phi} Z)$$

for all $f \in \mathcal{O}[\![Z]\!]$. 87

ψ_{Col}

$= \sigma_{K|L}^{-1} \circ \widetilde{\psi_{\mathrm{Col}}} = \widetilde{\psi_{\mathrm{Col}}} \circ \sigma_{K|L}^{-1}$. 88

$\psi_{K|L}$

$= \frac{1}{\pi_L}\phi_{K|L}^{-1} \circ \mathrm{Tr}$, where Tr is the trace map of $\mathscr{B}_{K|L}|\phi_{K|L}(\mathscr{B}_{K|L})$. 95

$\psi_{\mathcal{N}_{E|L}(V)}$

From $\psi_{\mathcal{M}_{\mathcal{K}|L}(V)}$ induced homomorphism from $\varphi^*\mathcal{N}_{\mathcal{K}|L}(V)$ to $\mathcal{N}_{\mathcal{K}|L}(V)$, where $V \in \mathbf{Rep}_L^{\mathrm{cris,an}}$ and $\mathcal{K}|L$ finite. 190

ψ_{S_∞}

inverse of the Frobenius φ_{S_∞} of S_∞. 186

$\mathbb{Q}_p$

Field of p-adic numbers. 51

$\overline{\mathbb{Q}_p}$

Fixed algebraic closure of $\mathbb{Q}_p$. 51

Q_ϕ

$= \frac{[\pi_L]_\phi \omega_\phi}{\omega_\phi}$. 173

rec

The map from $\varprojlim K_n^\times$ into the maximal abelian pro-p quptient $H_K^{\mathrm{ab}}(p)$ of H_K induced from the reciprocity map. 113

$\mathrm{rec}_{\mathbf{E}_K}$

The reciprocity homomorphism $\mathbf{E}_K \to H_K^{\mathrm{ab}}(p)$ in characteristic p. 113

$\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G)$

Category of finitely generated G-representations of $\mathcal{O}_L$, where G is a group. 74

$\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg,f})}(G)$

Category of finite free G-representations of $\mathcal{O}_L$, where G is a group. 169

$\mathbf{Rep}_L^{\mathrm{cris,an}}(G_L)$

Full subcategory of $\mathbf{Rep}_L^{(\mathrm{fg})}(G_L)$ consisting of the crystalline and analytic reprensentations. 173

$\mathbf{Rep}_{\mathcal{O}_L}^{\mathrm{cris,an}}(G_L)$

Full subcategory of $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg,f})}(G_L)$ consisting of the crystalline and analytic reprensentations. 173

Res

Residue homomorphism from $\Omega^1_{\mathscr{A}_{\mathcal{K}|L}}$ to $\mathcal{O}_{\mathcal{K}}$ with $\mathrm{Res}(\sum a_i Z^i \mathrm{d}Z) = a_{-1}$. 96

$\mathbf{R}\Gamma(\mathfrak{C}^\bullet)$

The image of the complex $\mathfrak{C}^\bullet$ in the derived corresponding category. 145

$\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, M)$

$= \mathbf{R}\Gamma(\mathcal{C}^\bullet(G, M))$ for a profinite group G and a topological G-module M. 145

$\mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(\mathcal{K}_\infty|\mathcal{K}, T)$

$= \mathbf{R}\Gamma_{\mathrm{cts}}(G_{\mathcal{K}}, \mathcal{F}_{\Gamma_{\mathcal{K}}}(T))$, where T is an $\mathcal{O}_L$-representation of $G_{\mathcal{K}}$ and $\mathcal{K}|L$ finite. 155

$\mathcal{R}_{\mathcal{K}}^{+}$

Subring of the power series ring with coefficients in $\mathcal{K}$, consisting of those power series converging for all $z \in \mathbb{C}_p$ with absolute value less than 1, $\mathcal{K}|L$ finite. 193

$\mathcal{R}_{\mathcal{K}}^{I}$

Ring inside $\mathcal{K}[\![Z]\!]$, consisting of those elements converging for $z \in \mathbb{C}_p$ with absolute value in I, where $I \subseteq [0,1]$ is an interval and $\mathcal{K}$ is a complete extension of L. 193

$\mathcal{R}_{\mathcal{K}}^{[r,1)}$

$= \varprojlim_{r<s<1} \mathcal{R}_{\mathcal{K}}^{[r,s]}$ with $0 < r < 1$ and where $\mathcal{K}$ is a complete extension of L. 193

$\mathcal{R}_{\mathcal{K}}$

$\cup_{0<r<1} \mathcal{R}_{\mathcal{K}}^{[r,1)}$, where $\mathcal{K}$ is a complete extension of L. 194

$R[\![X_1, \dots X_n]\!]$

Power series ring with variables $X_1, \dots X_n$ and with coefficients in the ring R. 52

$R((X_1, \dots, X_n))$

Ring of Laurent series in the variables $X_1, \dots, X_n$ with coefficients in R. 54

σ_{F_n}

generator of the Galois group $\Upsilon_n = \mathrm{Gal} F_n|F$. 170

$\sigma_{\mathcal{K}|L}$

Lift of the q_L-Frobenius of the residue class field extension to $\mathcal{K}$ for $\mathcal{K}|L$ finite unramified. 69

S_n

$= (\mathcal{O}_{F_n}[\Upsilon_n])^{\Delta_1 = \Delta_2}$. 180

S_∞

$= \varprojlim_n S_n$. 186

${}_f\otimes_R$

Tensor product over the ring R, where f is an endomorphism of R and the module on the left is considered as right R-module via f while the module on the right has its usual left-operation from R. 73

$\widehat{\otimes}$

Completed tensor product. 191

$\overset{\mathbf{L}}{\otimes}_R$

Tensor product in the derived category over the ring R. 155

$\mathcal{T}\mathcal{G}_\phi$

Tate module of $\mathcal{G}_\phi$. 52

Θ_n

Galois group of $F_\infty|F_n$. 187

$\Theta_{\mathcal{O}_{\mathbb{C}_p^\flat}}$

Surjective homomorphism from $W(\mathcal{O}_{\mathbb{C}_p^\flat})$ to $\mathcal{O}_{\mathbb{C}_p^\flat}$ with kernel generated by $\xi = \tau(\widetilde{\pi_L} - \pi_L)$. 171

t_{LT}

$= \log_{\mathrm{LT}}(\omega_\phi)$. 172

$\mathcal{TOP}_G$

Category of topological abelian Hausdorff groups with a continuous action from a profinite group G together with continuous group homomorphisms respecting the action from G. 23

$\mathcal{TOP}_{G,M}$

Category of topological abelian Hausdorff groups with commuting continuous actions f rom a profinite group G and a topological monoid M together with continuous group homomorphisms respecting the actions from G and M. 23

$\mathrm{Tot}(A^{\bullet,\bullet})$

Total complex of the double complex $A^{\bullet,\bullet}$. 30

$\mathrm{Tot}^n(A^{\bullet,\bullet})$

n-th object of the total complex $\mathrm{Tot}(A^{\bullet,\bullet})$ of the double complex $A^{\bullet,\bullet}$. 30

Tr_n

Trace map of $F_n|F_{n-1}$. 184

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